On the Complexity of Kings

Edith Hemaspaandra[1] Lane A. Hemaspaandra[1]
Till Tantau[2] Osamu Watanabe[3]

[1]University of Rochester, USA
[2]Universität zu Lübeck, Germany
[3]Tokyo Insititute of Technology, Japan

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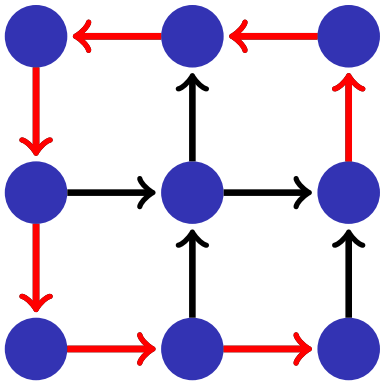
Outline

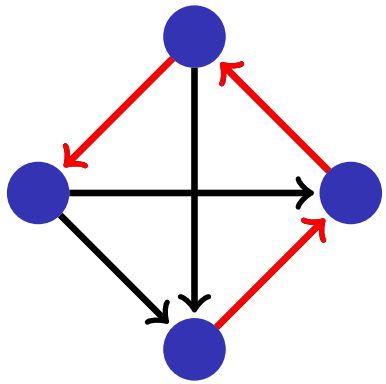
Two Examples of Graphs and Their Diameters

Definition

The diameter of a graph is the maximum distance of any two vertices in the graph.

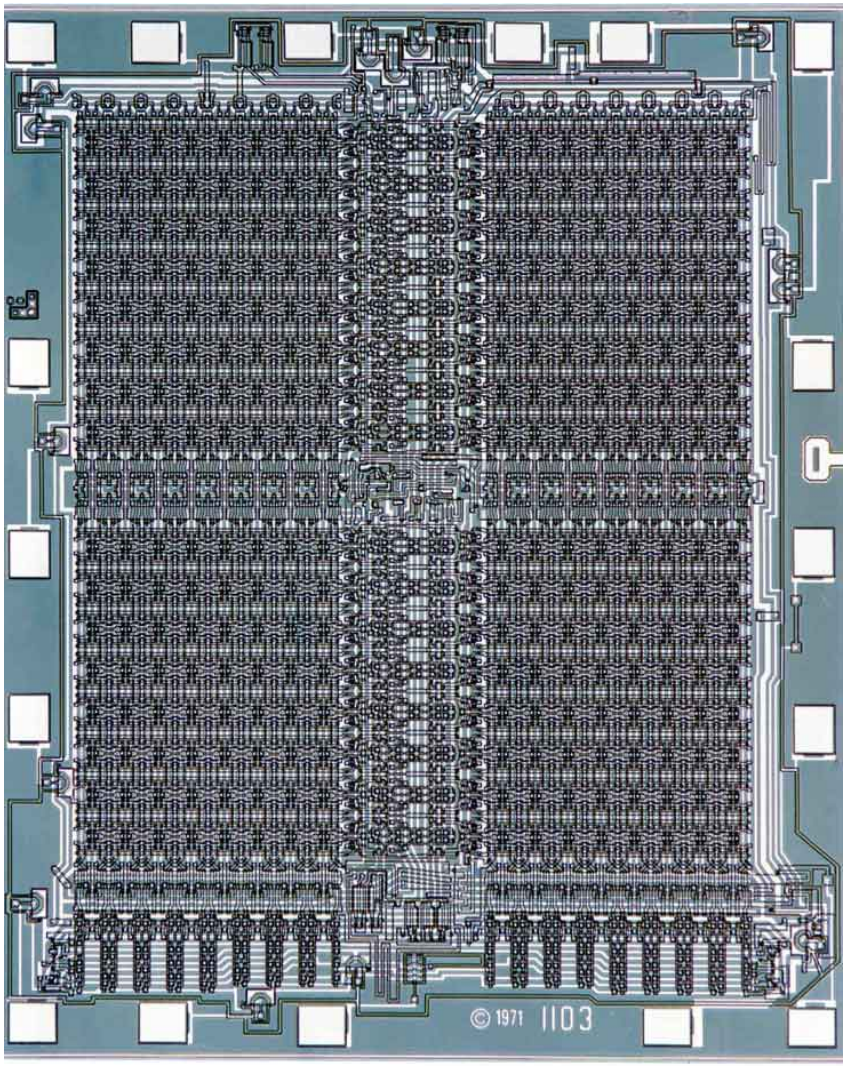


The diameter is 7.



The diameter is 3.

Really Large Graphs . . .

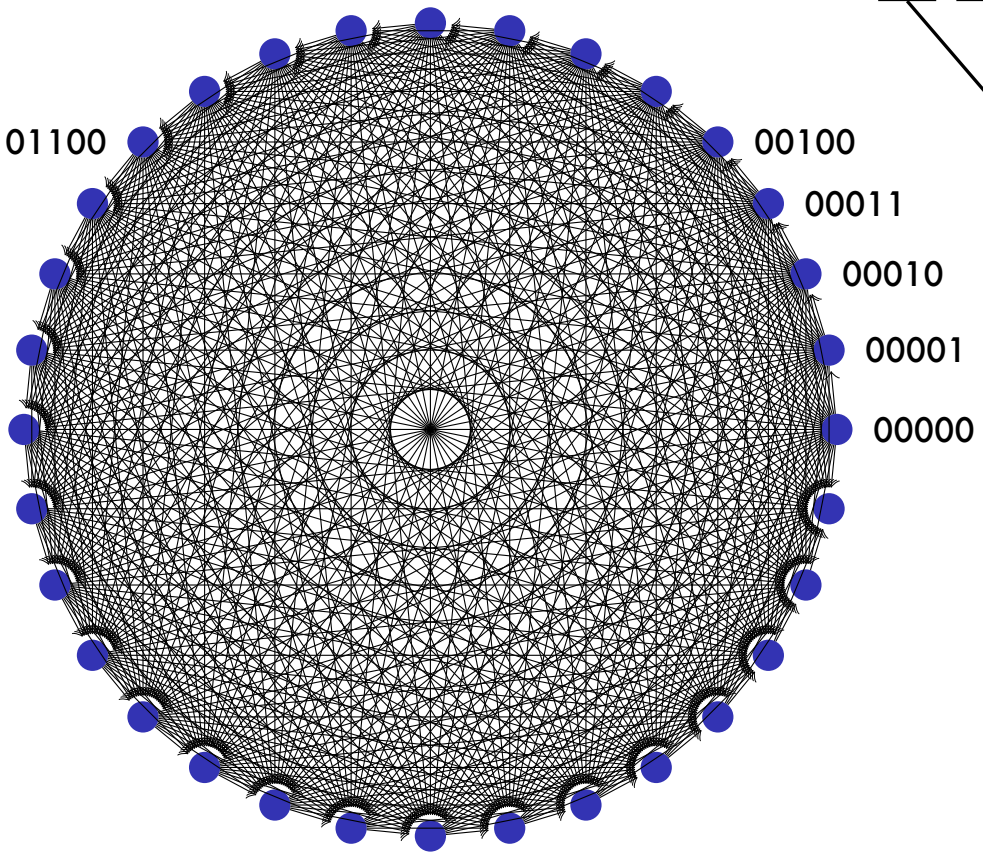


. . . Can Have Short Representation

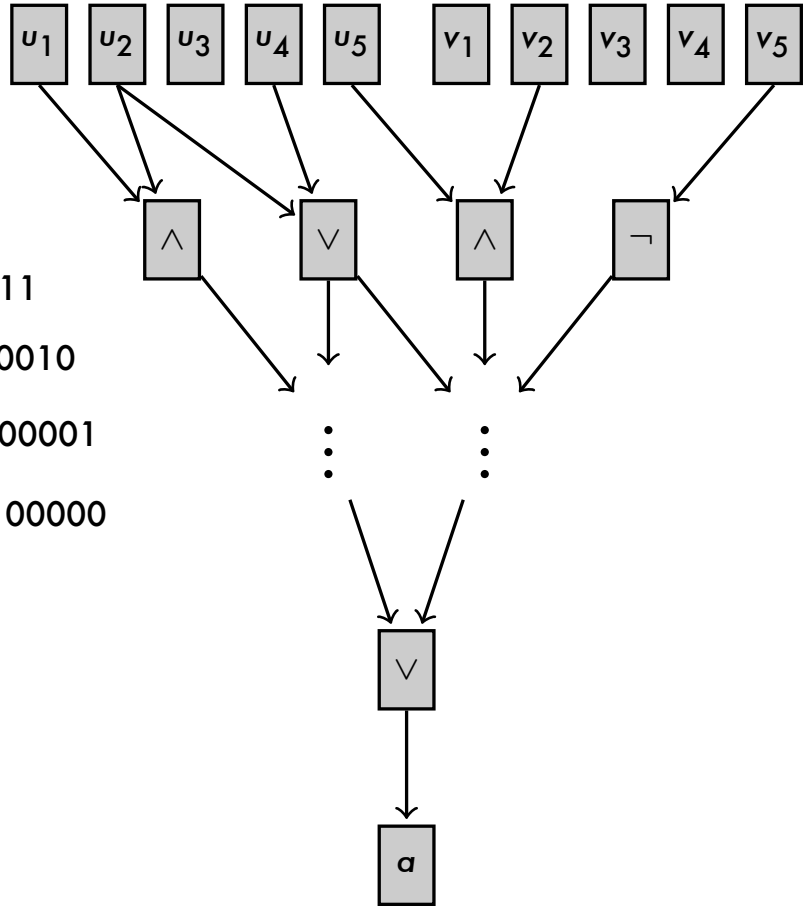
```
library IEEE;
use IEEE.std_logic_1164.all;
use work.std_arith.all;

entity COUNT is
  port( CLK, ENA : in std_logic;
        Q        : buffer std_logic_vector(31 downto 0);
        CARRY    : out std_logic);
end COUNT;

architecture A of COUNT is
begin
   P1: process( CLK )
   begin
      if( CLK'event and CLK = '1' ) then
         if( ENA = '1' ) then
            Q <= Q + 1;
         end if;
      end if;
   end process;
   CARRY <= '1' when Q = "1111" else '0';
end A;
```

Galperin–Wigderson Model of Succinctly Represented Graphs

Graph with 2^n vertices . . .



. . . and its representing circuit



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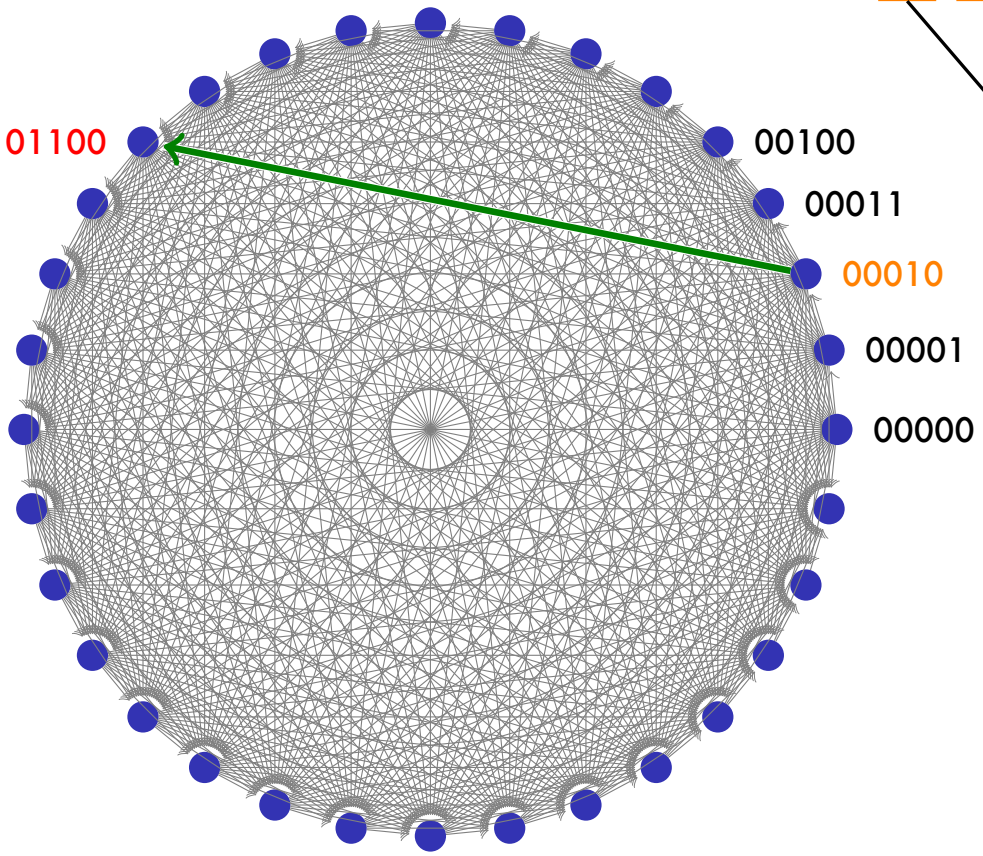

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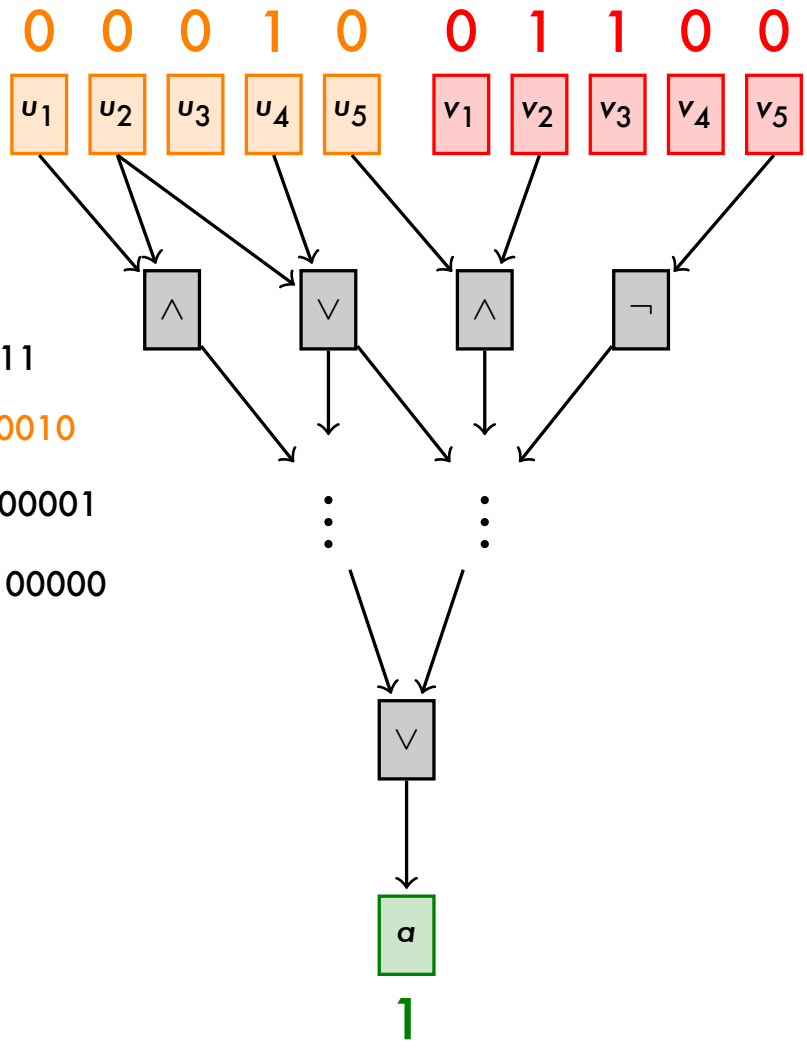


Definition of the Succinct k-Diameter Problem.

Definition (Succinct k-Diameter Problem)

For $k \geq 2$ the problem succinct-k-diameter is:

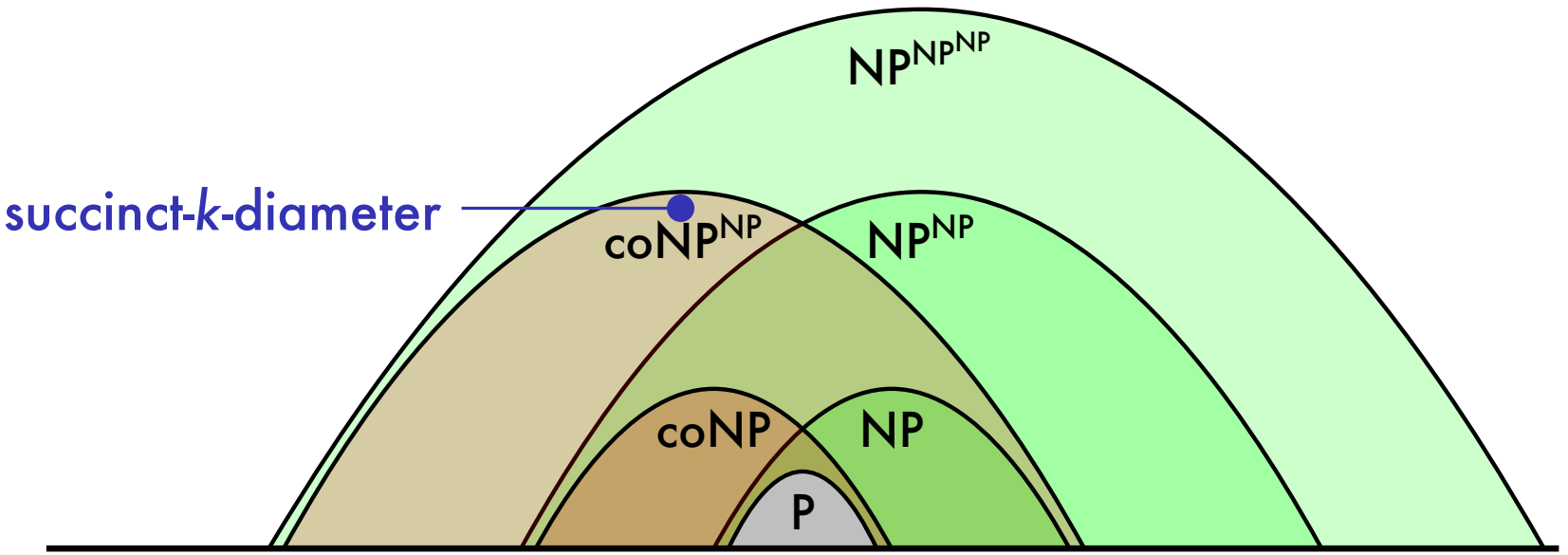
Input A circuit C with $2n$ inputs and 1 output.

Question Does the 2^n-vertex graph G represented by C have diameter at most k?

The Succinct k-Diameter Problem is Complete for the Second Level of the Polynomial Hierarchy.

Theorem

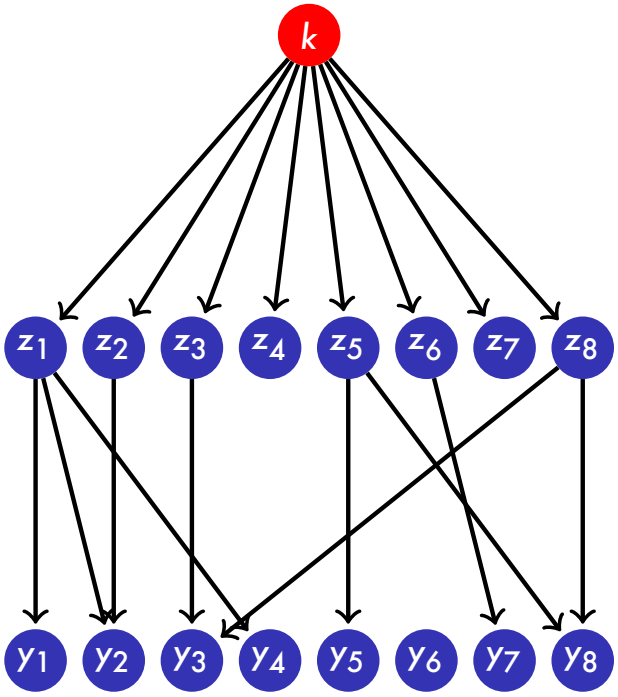
Let $k \geq 2$. Then succinct-k-diameter is complete for $\mathrm{coNP}^{\mathrm{NP}}$.

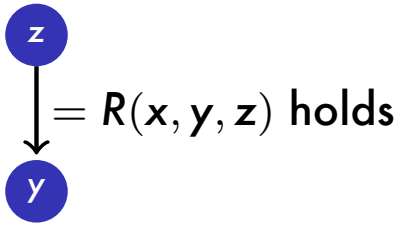


The Proof Ideas.

Reduction of $L \in \mathrm{coNP}^{\mathrm{NP}}$ to succinct-2-diameter.

1. By the Stockmeyer characterization there is a simple predicate R with $x \in L \iff \forall y \exists z \colon R(x, y, z)$
2. From k, we can reach all y in 2 steps iff $\forall y \exists z \colon R(x, y, z)$.
3. From $v \neq k$, we can reach all vertices in 2 steps.





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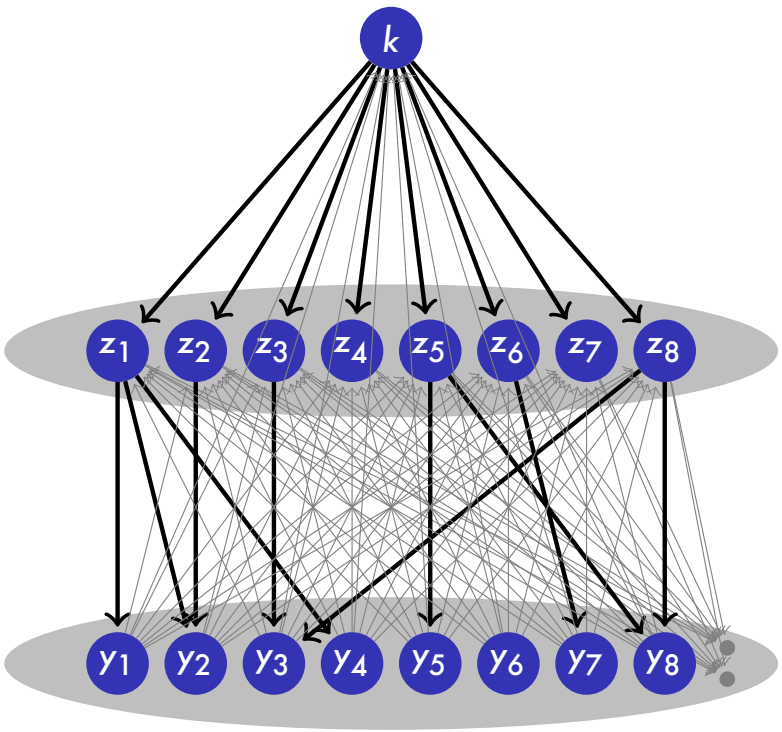


all missing arrows point upward

= Diameter 2 tournament

z

= $R(x, y, z)$ holds

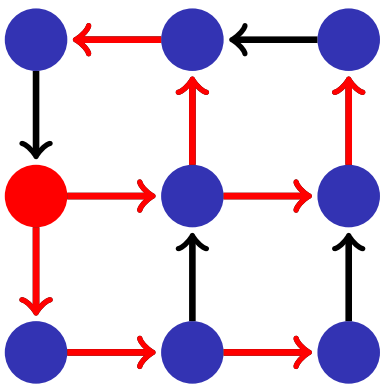
y

= Diameter 2 tournament

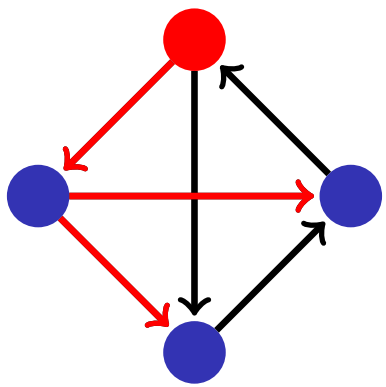
Two Examples of Graphs, Revisited, and Their Radius

Definition

- A ball of radius k is the set of vertices that are reachable in k steps from the center of the ball.
- The radius of a graph is the smallest size of a ball that covers the whole graph.



The radius is 3.

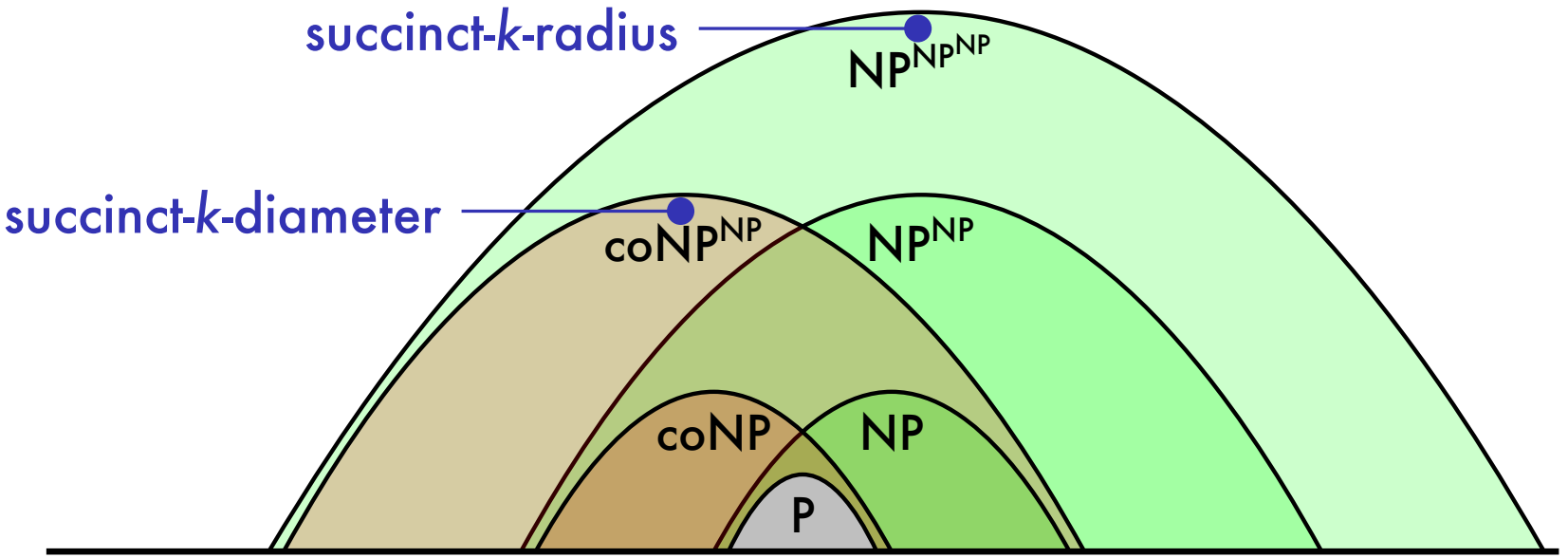


The radius is 2.

The Succinct k-Radius Problem is Complete for the Third Level of the Polynomial Hierarchy.

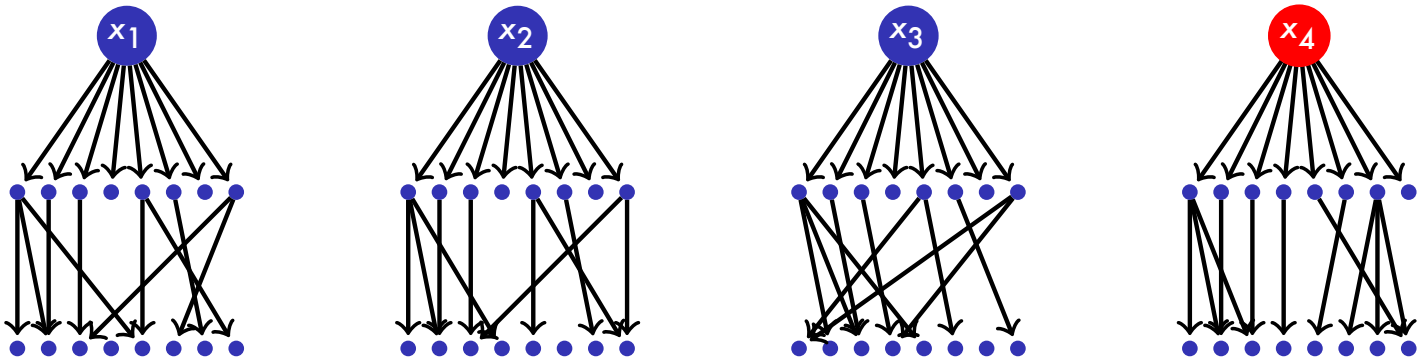
Theorem

Let $k \geq 2$. Then succinct-k-radius is complete for $\mathrm{NP}^{\mathrm{NP}^{\mathrm{NP}}}$.



The Proof Ideas.
Reduction of $L \in \mathrm{NP}^{\mathrm{NP}^{\mathrm{NP}}}$ to succinct-2-radius.

1. By the Stockmeyer characterization there is a simple predicate R with $w \in L \iff \exists x \forall y \exists z \colon R(w, x, y, z)$
2. From x_i, we reach all its y in 2 steps iff $\forall y \exists z \colon R(w, x_i, y, z)$.
3. From x_i, we reach all other vertices in 2 steps.



The Proof Ideas.
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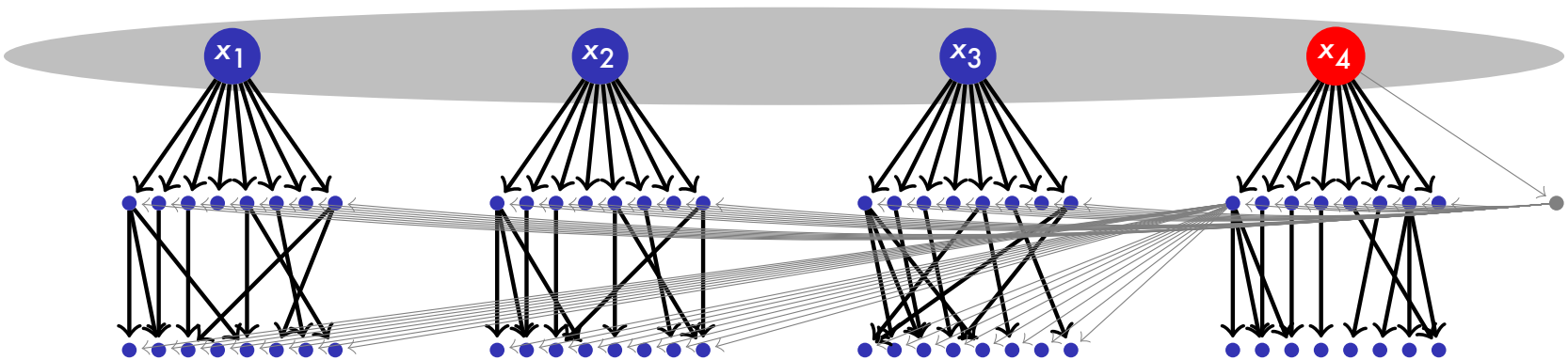
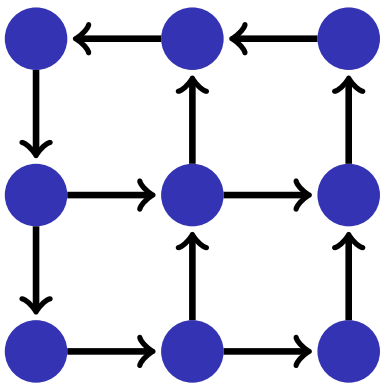


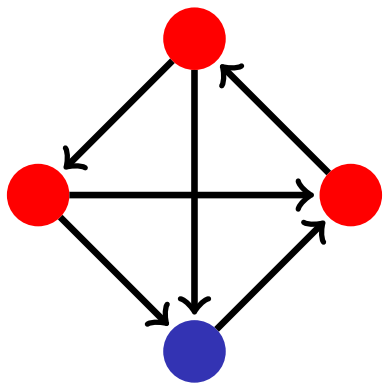
Two Examples of Graphs, Re-Revisited

Definition

A *k*-king of a graph is a vertex from which all vertices can be reached in at most *k* steps.



The 2-kings.



The 2-kings.

Trivia On Kings.

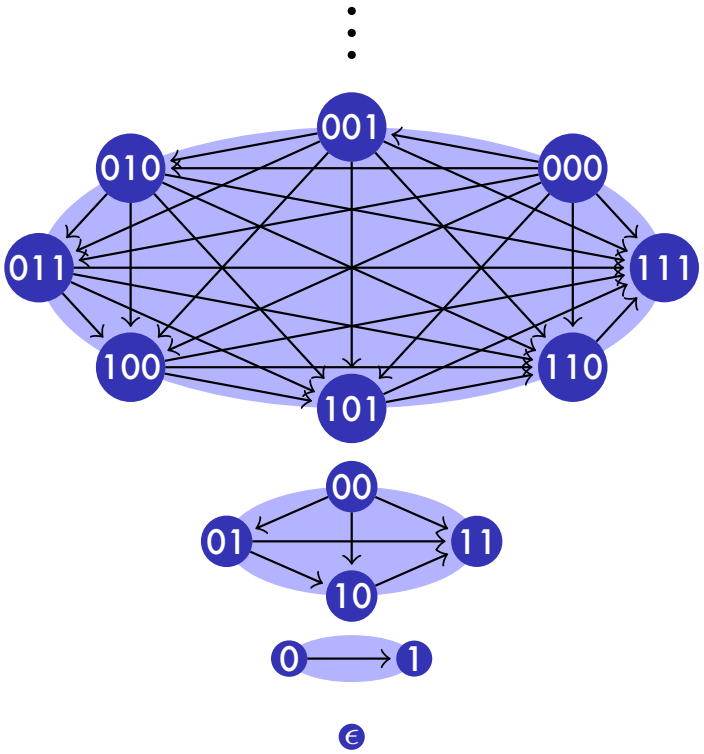
- Landau has studied lions and noted that a pride of lions forms a tournament and always has a 2-(lion)-king.
- A graph has radius k iff there exists a k-king.
- A graph has diameter k iff all vertices are k-kings.

Author Rick Marin, public domain

We Tie Our Hands.

Definition

A uniform tournament family consists of one tournament per word length. Each T_i is succinctly specified by a circuit C_i from a P-uniform circuit family $(C_i)_{i \in \mathbb{N}}$.



$= T_3$, specified by C_3

$= T_2$, specified by C_2

$= T_1$, specified by C_1

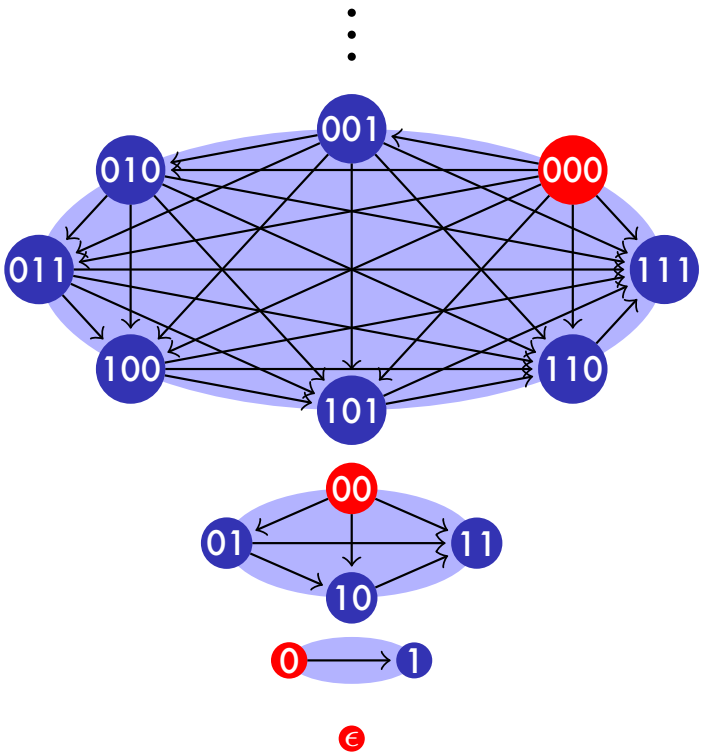
$= T_0$, specified by C_0

Definition of King Languages.

Definition

The set of k-kings in the tournaments of a uniform tournament family forms a k-king language.

(Remark: They are closely related to P-selective sets.)

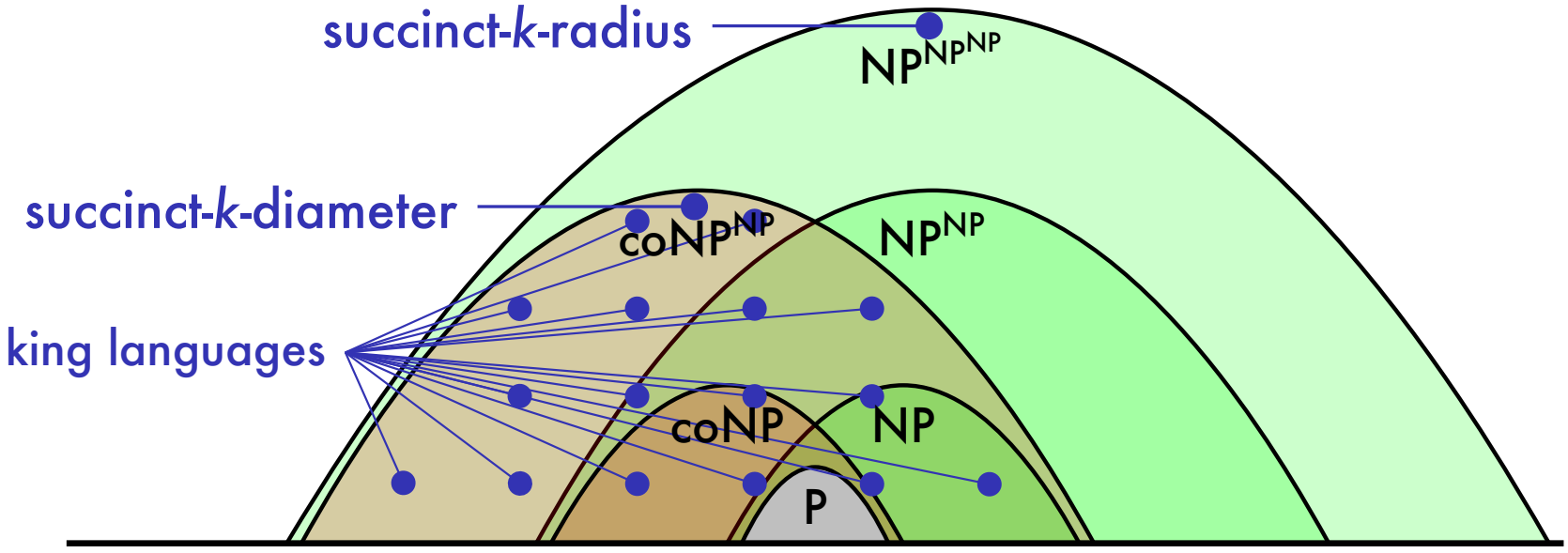


$L = \{\epsilon, 0, 00, 000, \dots\}$
is a 2-king language

The Main Result: All Languages in $\mathrm{coNP}^{\mathrm{NP}}$ Are King Languages in Disguise.

Theorem

$L \in \mathrm{coNP}^{\mathrm{NP}}$ holds iff L is equivalent to a king language.



Summary

Let $k \geq 2$.

- succinct-k-diameter is complete for $\mathrm{coNP}^{\mathrm{NP}}$.
- succinct k-radius is complete for $\mathrm{NP}^{\mathrm{NP}^{\mathrm{NP}}}$.
- $L \in \mathrm{coNP}^{\mathrm{NP}}$ iff L is equivalent to a k-king language.