UNIVERSITÄT ZU LÜBECK
INSTITUTE FOR
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Natural Complete Problems for Parametrized Space Classes

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IM FOCUS DAS LEBEN

Talk Outline

Introduction: Parametrized Space

Limited Nondeterminism

- The Concept
- The Classes
- Natural Complete Problems

Limited Time and Space

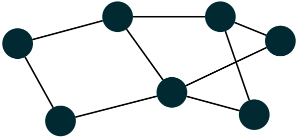
- The Concept
- The Classes
- Natural Complete Problems

Parametrized Complexity in a Nutshell.

The Vertex Cover Problem

Given a graph with n vertices, can we choose k of them so that all edges are "covered"?

Example (A Graph with a Vertex Cover of Size 3)



- The vertex cover problem is NP-complete and can easily be solved in time 2^n and n^k.
- It can also be solved in time $2^k n$.
- This is called *fixed parameter tractable* since for *each fixed k* we get a linear time algorithm.

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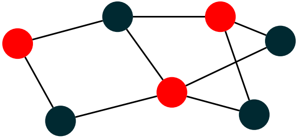

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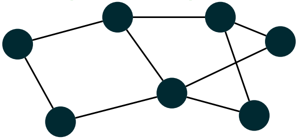
Parametrized *Space* Complexity in a Nutshell.

How much *space* is needed to solve the vertex cover problem?

Example (A Graph with a Vertex Cover of Size 3)



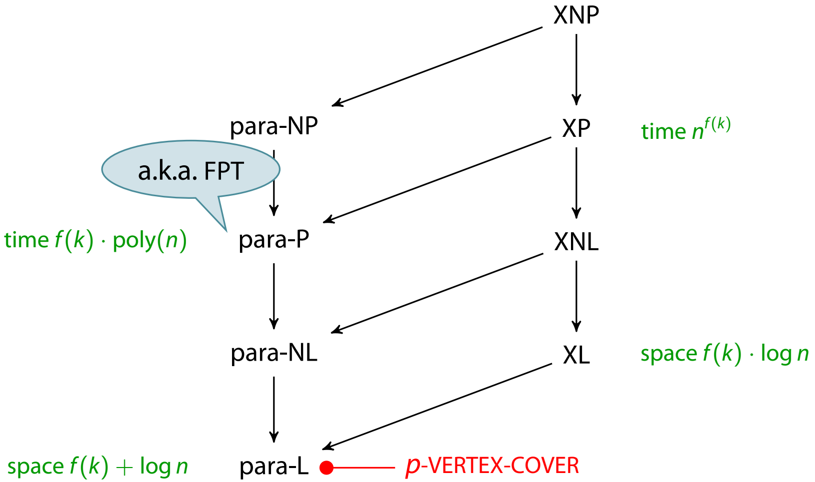
- The problem can easily be solved in space $O(k \log n)$.
- It can also be solved in space $O(k^2 + \log n)$.
- This is the *logspace analogue* of fixed parameter tractability.

Basic Classes

$$
\begin{tikzcd}
 & & \mathrm{XNP} \arrow[dl] \arrow[d] & \\
 & \text{para-NP} \arrow[d] & \mathrm{XP} \arrow[dl] \arrow[d] & \text{time } n^{f(k)} \\
\text{time } f(k)\cdot\mathrm{poly}(n) & \text{para-P} \arrow[d] & \mathrm{XNL} \arrow[dl] \arrow[d] & \\
 & \text{para-NL} \arrow[d] & \mathrm{XL} \arrow[dl] & \text{space } f(k)\cdot\log n \\
\text{space } f(k)+\log n & \text{para-L} & &
\end{tikzcd}
$$

a.k.a. FPT

Basic Classes



Talk Outline

Introduction: Parametrized Space

Limited Nondeterminism

- The Concept
- The Classes
- Natural Complete Problems

Limited Time and Space

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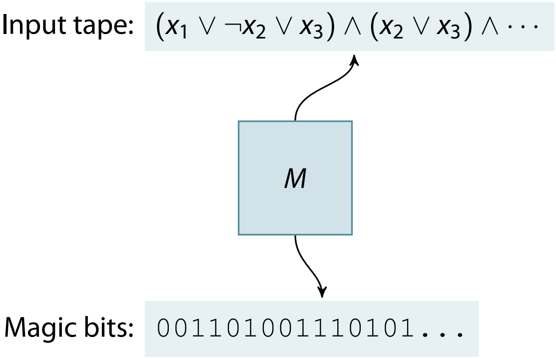
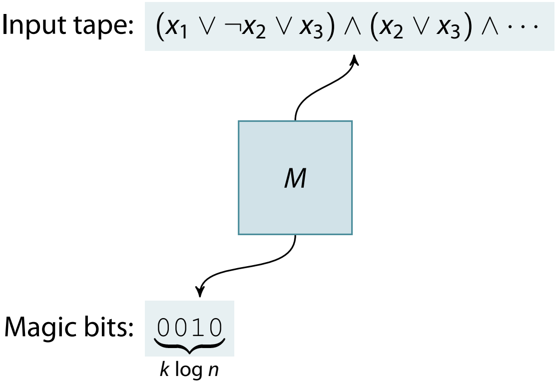
Limited Nondeterminism: The Concept

- Nondeterminism = determinism + magical bits.



Limited Nondeterminism: The Concept

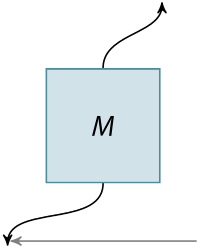
- Nondeterminism = determinism + magical bits.
- *Limited nondeterminism* = determinism + *few* magical bits.



Limited Nondeterminism: The Concept

- Nondeterminism = determinism + magical bits.
- *Limited nondeterminism* = determinism + *few* magical bits.

Input tape: $(x_1 \vee \neg x_2 \vee x_3) \wedge (x_2 \vee x_3) \wedge \cdots$



Magic bits: $\underbrace{\texttt{0010}}_{k \log n}$

Types of access:

- 1-way: β-classes
- 2-way: Σ-classes

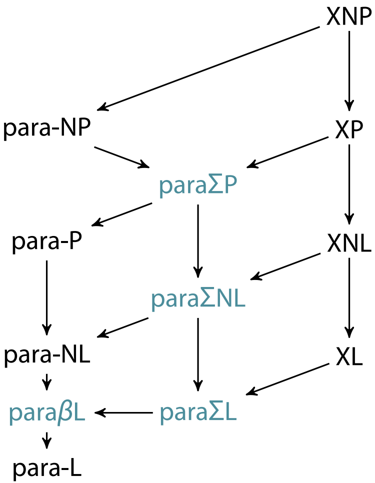
Parametrized Classes of Bounded Nondeterminism.

Definition

From a parametrized class para-C we derive the classes

1. paraβC by giving the para-C-machine 1-way access to $O(f(k) \log n)$ nondeterministic bits,
2. paraΣC by giving the para-C-machine 2-way access to $O(f(k) \log n)$ nondeterministic bits.

Parametrized Classes of Bounded Nondeterminism.



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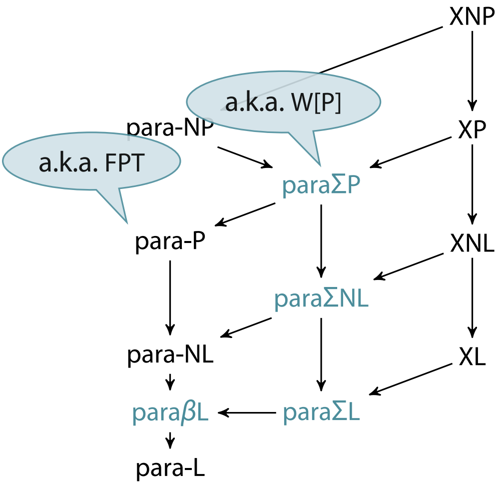
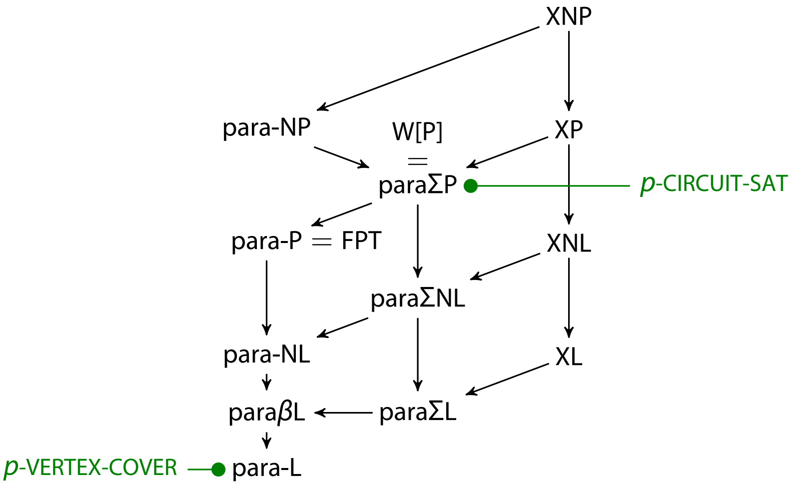


Known Complete Problems Under Para-L-Reductions.



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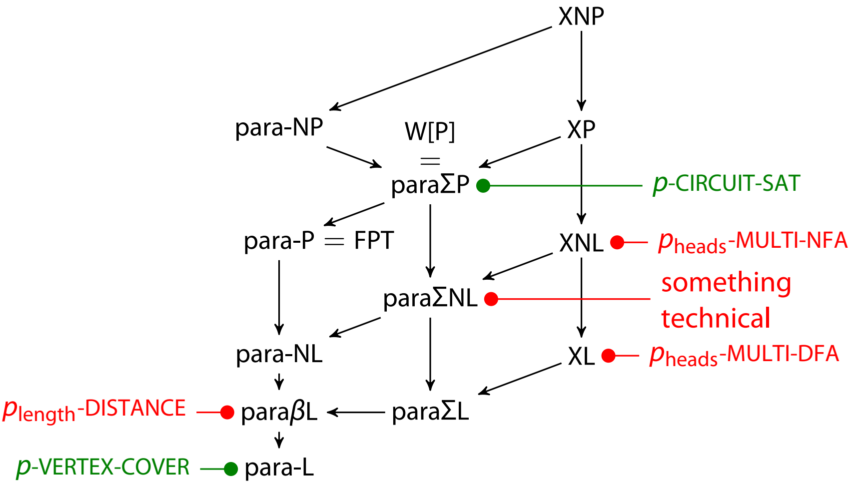

The Generability Problem.

The Problems GEN and AGEN

Input A table describing a function $\circ\colon U \times U \to U$ and a set $G \subseteq U$.

Question Is U the smallest superset of G that is closed under $\circ$?

For AGEN, $\circ$ must be associative (a semi-group).

Example

Does $G = \{0, 3\}$ generate $U = \{0, 1, 2, 3\}$ for

$\circ$	0	1	2	3
0	0	0	0	0
1	0	1	2	3
2	0	2	0	2
3	0	3	2	1

?

The Generability Problem.

The Problems p-GEN and p-AGEN

Input A table describing a function $\circ \colon U \times U \to U$ and a number k.

Question Is there a G of size k such that U is the smallest superset of G that is closed under $\circ$?

For p-AGEN, $\circ$ must be associative (a semi-group).

Example

Does

$\circ$	0	1	2	3
0	0	0	0	0
1	0	1	2	3
2	0	2	0	2
3	0	3	2	1

have a generating set G of size 2?

The Complexity of the Generability Problems.

Fact

GEN *is* P-*complete.*

Fact

p-GEN *is* paraΣP-*complete (= is* W[P]-*complete)*.

Fact

AGEN *is* NL-*complete.*

The Complexity of the Generability Problems.

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Theorem

p-AGEN *is* paraΣNL-*complete.*

The Complexity of the Generability Problems.

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Fact

AGEN *is* NL-*complete.*

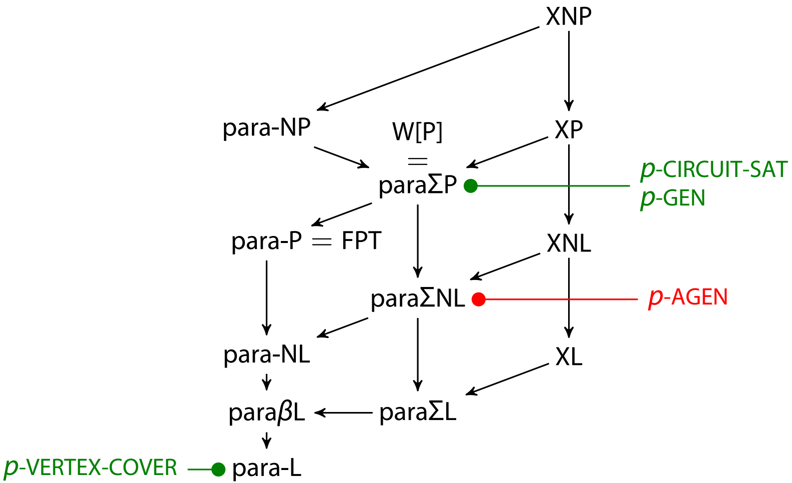
Theorem

p-AGEN *is* paraΣNL-*complete.*

Proof.

Highly technical reduction from "something technical." □

Known Complete Problems Under Para-L-Reductions.



Talk Outline

Limiting Time and Space Simultaneously: The Concept

Classical Complexity Theory

We limit *either* time *or* space, but not both because of the well-known inclusion chain

$$\mathrm{L} \subseteq \mathrm{NL} \subseteq \mathrm{P} \subseteq \mathrm{NP} \subseteq \mathrm{PSPACE} = \mathrm{NPSPACE}.$$

Parametrized Complexity Theory

- Inside the X-classes and the para-classes, analogous inclusions hold.
- *However,* new classes arise when we “mix” time X-classes and space para-classes or vice versa.

Limiting Time and Space Simultaneously: The Classes

Definition

Let t be a time bound and s a space bound that may depend on a parameter.

1. The class $\mathrm{D}[t, s]$ contains all parametrized problems decidable by a DTM working in *time t* and *space s*.
2. The class $\mathrm{N}[t, s]$ is the nondeterministic version of it.

Examples

- $\mathrm{D}[f\ \mathrm{poly}, \infty] = \text{para-P} = \mathrm{FPT}$.
- $\mathrm{D}[\infty, f \log] = \mathrm{XL}$.
- $\mathrm{D}[f\ \mathrm{poly}, f \log]$ is "FPT time, but using only XL space".

Simultaneous Time-Space Classes.

$$
\begin{tikzcd}[column sep=small]
\mathrm{D}[\infty, f\,\mathrm{poly}] = \text{para-PSPACE} \arrow[dd] \arrow[dr] \arrow[dddr] & & \mathrm{XNP} = \mathrm{N}[f\,\mathrm{poly}, \infty] \arrow[dd] \arrow[dl] \\
& \mathrm{N}[n^f, f\,\mathrm{poly}] \arrow[dl] & \\
\mathrm{N}[f\,\mathrm{poly}, \infty] = \text{para-NP} \arrow[dd] \arrow[dddr] & & \mathrm{XP} = \mathrm{D}[f\,\mathrm{poly}, \infty] \arrow[dd] \arrow[dl] \\
& \mathrm{D}[n^f, f\,\mathrm{poly}] \arrow[dl] & \\
\mathrm{D}[f\,\mathrm{poly}, \infty] = \text{para-P} \arrow[dd] \arrow[dddr] & & \mathrm{XNL} = \mathrm{N}[\infty, f\,\log] \arrow[dd] \arrow[dl] \\
& \mathrm{N}[f\,\mathrm{poly}, f\,\log] \arrow[dl] & \\
\mathrm{N}[\infty, f + \log] = \text{para-NL} \arrow[dd] & & \mathrm{XL} = \mathrm{D}[\infty, f\,\log] \arrow[dl] \\
& \mathrm{D}[f\,\mathrm{poly}, f\,\log] \arrow[dl] & \\
\mathrm{D}[\infty, f + \log] = \text{para-L} & &
\end{tikzcd}
$$

The Key Problem

The Longest Common Substring Problem p_{size}-LCS

Input	An alphabet Σ, a set S of strings from Σ^*, and a length l.
Parameter	$\lvert S \rvert$
Question	Is there a string s of length l that is a common substring of all elements of S?

Example

A longest common substring of `abbacd`, `badbab`, `cbacba` is `aba`.

LCS is Complete for A Time--Space Class.

Theorem

p_{size}-LCS *is complete for* $\mathrm{N}[f\ \mathrm{poly}, f\ \log]$.

LCS is Complete for A Time--Space Class.

Theorem

p_{size}-LCS *is complete for* $\mathrm{N}[f\,\text{poly}, f\,\log]$.

Proof.

Inclusion While guessing s, we only need $k \log n$ bits to keep track of the positions.

Hardness Technical chain of reductions via a parametrized version of cellular automata.

□

Complete Problems for Simultaneous Time-Space Classes.

$D[\infty, f\ \mathrm{poly}] = \text{para-PSPACE}$ $\qquad$ $\mathrm{XNP} = N[f\ \mathrm{poly}, \infty]$

$N[n^f, f\ \mathrm{poly}]$

$N[f\ \mathrm{poly}, \infty] = \text{para-NP}$ $\qquad$ $\mathrm{XP} = D[f\ \mathrm{poly}, \infty]$

$D[n^f, f\ \mathrm{poly}]$

$D[f\ \mathrm{poly}, \infty] = \text{para-P}$ $\qquad$ $\mathrm{XNL} = N[\infty, f \log]$

$N[f\ \mathrm{poly}, f \log]$ — p_{size}-LCS

$N[\infty, f + \log] = \text{para-NL}$ $\qquad$ $\mathrm{XL} = D[\infty, f \log]$

$D[f\ \mathrm{poly}, f \log]$ — p_{size}-GREEDY-LCS

$D[\infty, f + \log] = \text{para-L}$

Summary

- Limited nondeterminism allows us to define natural analogues of W[P] for parameterized space.
- The associative generability problem is complete for the NL-analogue of W[P].
- The longest common subsequence problem parametrized by the number of strings is complete for "nondeterministic simultaneous FPT time and XL space."