

Algorithmic Metatheorems 2.0 Till Tantau

Theorietag in Speyer, September 2015

IM FOCUS DAS LEBEN



Outline

Algorithmic Metatheorems

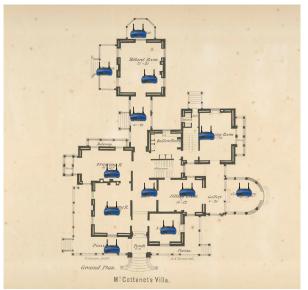
- Classic Variants ...
- …and New Variants

New Variants 1: Space Complexity

- New Theorems
- Applications: Cycle Lengths in Graphs
- Applications: Quantifier Prefix Classes
- Applications: Integer Optimization

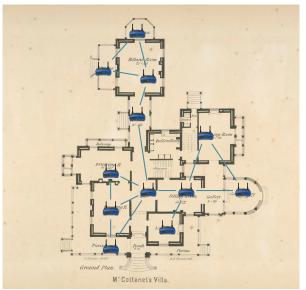
New Variants 2: Circuit Complexity

- New Theorems
- Applications: Visible Pushdown Languages



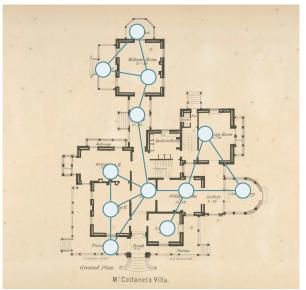
We install WLAN routers in a home.

Public Domain

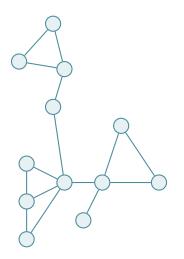


- We install WLAN routers in a home.
- Adjacent routers interfere.

Public Domain



- We install WLAN routers in a home.
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- We install WLAN routers in a home.
- Adjacent routers interfere.
- The abstract problem is of course
 3-colorability.

- 3-Colorability is a classical NP-complete problem.
- Assuming $P \neq NP$, it *cannot* be solved efficiently.

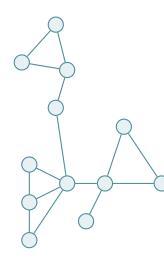
However

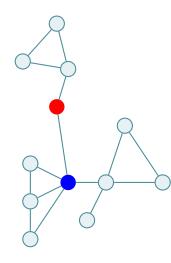
What happens, when the input has a *tree-like decomposition* and we can apply *divide and conquer*?

Entzwei und gebiete! Tüchtig Wort. – Verein und leite! Besserer Hort.

Johann Wolfgang Goethe

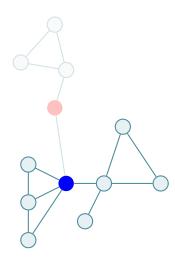




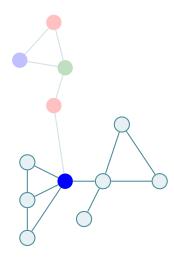


Sometimes divide and conquer can be applied to 3-colorability.

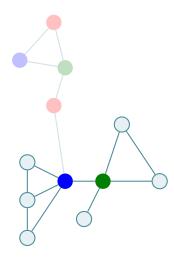
Pick two appropriate vertices and consider the 6 possible colorings.



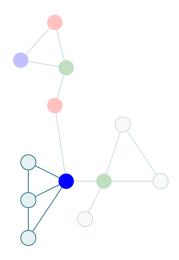
- Pick two appropriate vertices and consider the 6 possible colorings.
- A vertex *screens* part of the graph



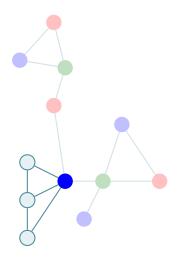
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- and we can solve the problem recursively on the independent parts.



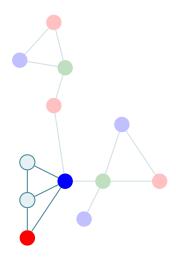
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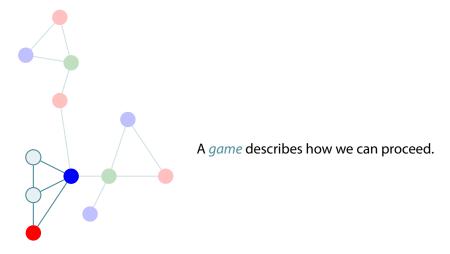
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- Pick two appropriate vertices and consider the 6 possible colorings.
- A vertex *screens* part of the graph
- and we can solve the problem recursively on the independent parts.
- It can happen that we have to *remember several vertices* during the recursion.

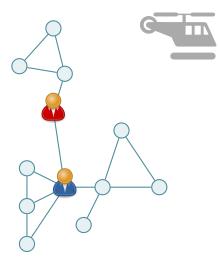


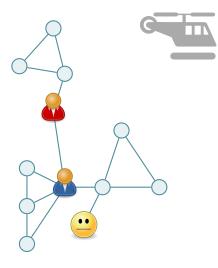
Game Objective

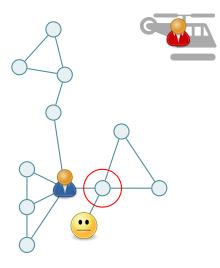
The *k* cops try to catch a robber by being on the same vertex as her.

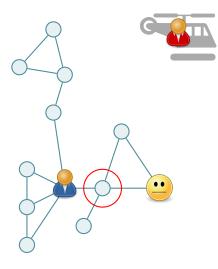
Game Rules

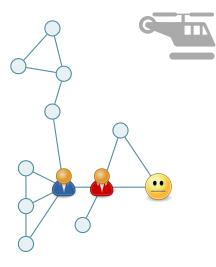
- 1. First the cops, then the robber pick a start vertex.
- 2. A cop gets on a *helicopter* and heads towards some vertex.
- 3. *Meanwhile* the robber moves along *unoccupied vertices*.

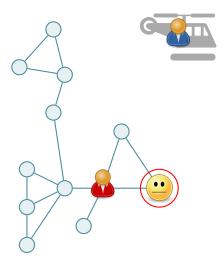


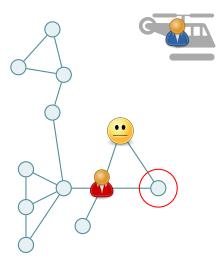


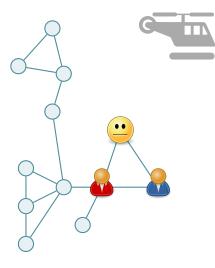


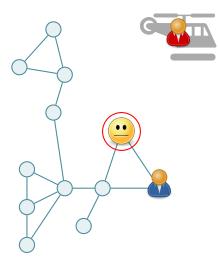


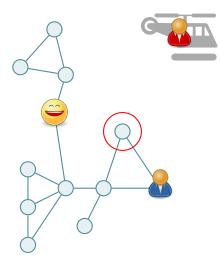


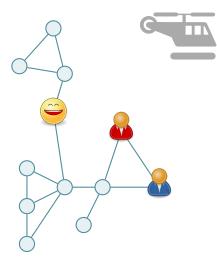


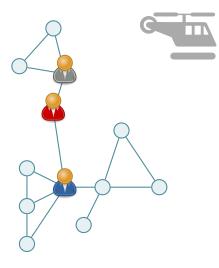


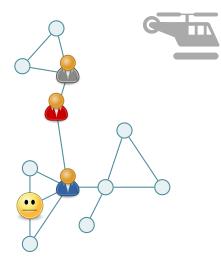


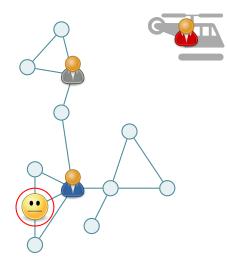


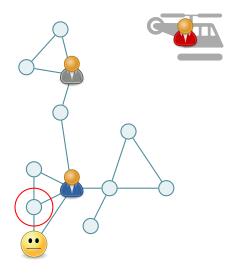


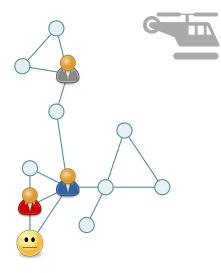


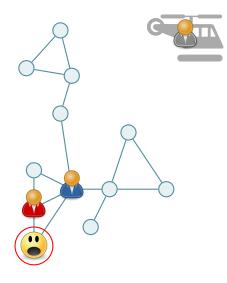


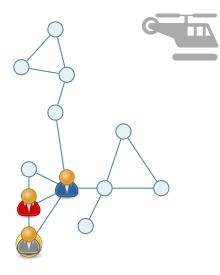










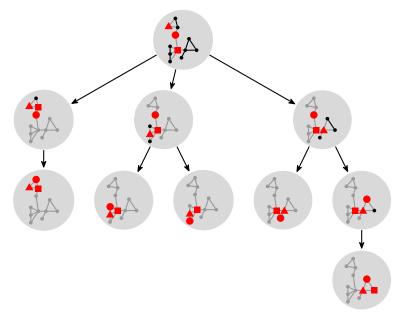


The Cops's Strategy = Tree Decomposition

The winning strategy of the cops forms a tree:

- The *nodes* are positions of the cops.
- The *root* is their initial position.
- The children of a tree node are the graph components that could contain the robber.

Example of a Tree Decomposition of Width 2.



Divide and Conquer, Second Attempt

Theorem

For a graph G let a tree decomposition of width k with n nodes be given. Then we can decide in time

 $O(3^{k+1}n)$

whether G is 3-colorable.

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whether G is 3-colorable.

Theorem

For a graph G let a tree decomposition of width k with n nodes be given. Then we can compute in time

 $O(2^{k+1}n)$

the largest independent set of G.

Why We Need *Meta* theorems.

Theorem 4.4

Each of the following problems is in NC, when restricted to graphs with treewidth $\leq K$, for constant K: vertex cover [GT1], dominating set [GT2], domatic number [GT3], chromatic number [GT4], monochromatic triangle [GT5], feedback vertex set [GT7], feedback arc set [GT8], partial feedback edge set [GT9], minimum maximal matching [GT10], partition into triangles [GT11], partition into isomorphic subgraphs for fixed H [GT12], partition into Hamiltonian subgraphs [GT13], partition into forests [GT14], partition into cliques [GT15], partition into perfect matchings [GT16], clique [GT19], independent set [GT20], induced subgraph with property P (for monadic second order properties P) [GT21], induced connected subgraph with property P (for monadic second order properties P) [GT22], induced path [GT23], balanced complete bipartite subgraph [GT24], bipartite subgraph [GT25], degree bounded connected subgraph for fixed d [GT26], planar subgraph [GT27], transitive subgraph [GT29], uniconnected subgraph [GT30], minimum k-connected subgraph for fixed k [GT31], cubic subgraph [GT32], minimum equivalent digraph [GT33], Hamiltonian completion [GT34], Hamiltonian circuit [GT37], directed Hamiltonian circuit [GT38], Hamiltonian path (and directed Hamiltonian path) [GT39], subgraph isomorphism for fixed H, subgraph isomorphism for connected H with bounded valence [GT 48], graph contractability for fixed H [GT51], graph homomorphism for fixed H [GT52], path with forbidden pairs for fixed n [GT54], multiple choice matching for fixed J [GT55], graph grundy numbering for graphs with bounded valence [GT56], kernel [GT57], k-closure [GT58], path distinguishers [GT60], degree constrained spanning tree [ND1], maximum leaf spanning tree [ND2], bounded diameter spanning tree [ND3], k'th best spanning tree for fixed k [ND9], bounded component spanning forest for fixed k [ND10], multiple choice branching for fixed m [ND11]. Steiner tree in graphs [ND12], max cut [ND16], minimum cut into bounded sets [ND17], rural postman [ND27], longest circuit [ND28], longest path [ND29], shortest weight-constrained path [ND30], k'th shortest path for fixed k [ND31], disjoint connecting paths for fixed k [ND40], maximum length-bounded disjoint paths for fixed J [ND41], maximum fixed-length disjoint paths for fixed J [ND42], chordal graph completion for fixed k, chromatix index, spanning tree parity problem, distance d chromatic number for fixed d and k, thickness $\leq k$ for fixed k, membership for each class C of graphs, which is closed unded minor taking.

Outline

Algorithmic Metatheorems

- Classic Variants ...
- ...and New Variants

New Variants 1: Space Complexity

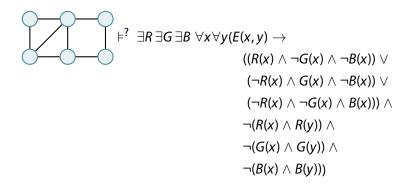
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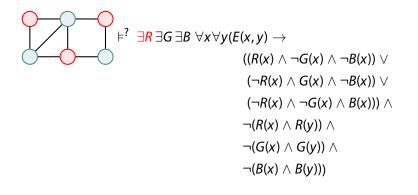
New Variants 2: Circuit Complexity

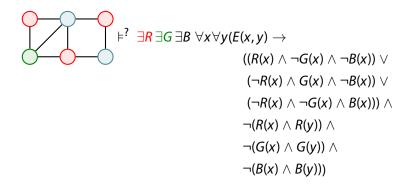
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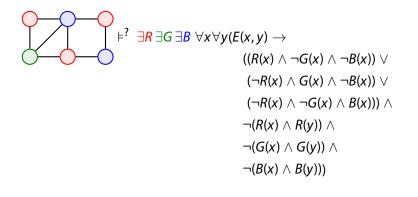
All Problems Share one Property.

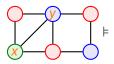
In 1990, Courcelle noted that all of the problems can be described in *monadic second order logic* (MSO logic).



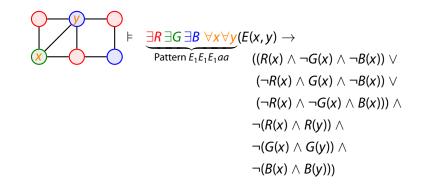








 $\exists R \exists G \exists B \forall x \forall y (E(x, y) \rightarrow ((R(x) \land \neg G(x) \land \neg B(x))) \lor (\neg R(x) \land G(x) \land \neg B(x))) \lor (\neg R(x) \land \neg G(x) \land B(x))) \land \neg (R(x) \land \neg G(x) \land B(x))) \land \neg (G(x) \land G(y)) \land \neg (G(x) \land G(y)) \land \neg (B(x) \land B(y)))$

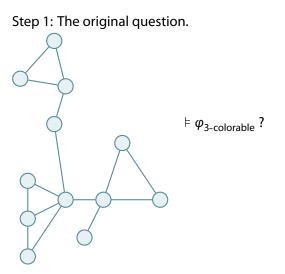


Theorem (Courcelle)

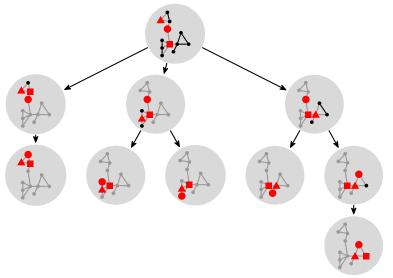
Let φ be an MSO formula and k a number. Then

$\{G \mid G \models \varphi \text{ and } G \text{ has tree width at most } k\}$

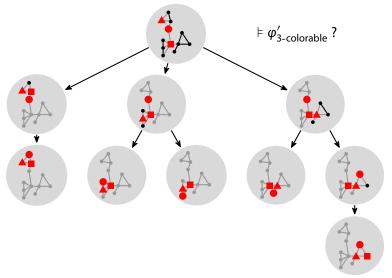
can be decided in linear time.



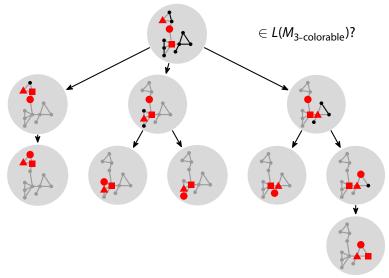
Step 2: Compute a tree decomposition in linear time.



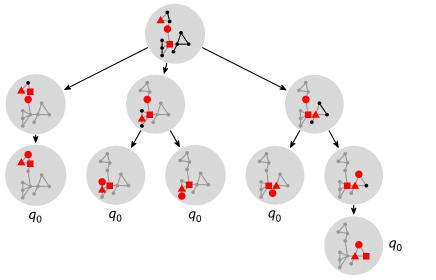
Step 3: Adjust the formula so that it applies to the tree.



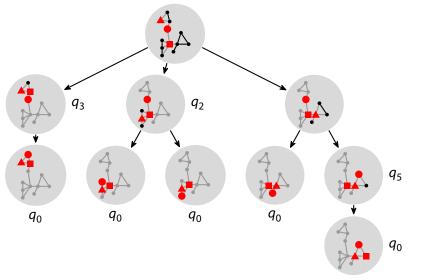
Step 4: Transform the formula into a tree automaton.



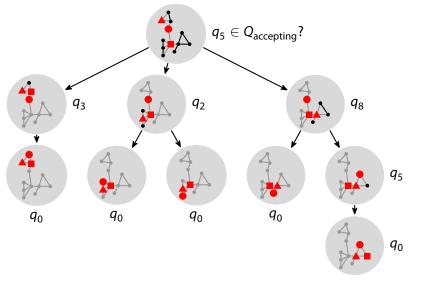
Step 5: Evaluate that automaton in linear time.



Step 5: Evaluate that automaton in linear time.



Step 5: Evaluate that automaton in linear time.



Theorem (The General Pattern)

lf

- the problem can be described in a certain logic
- and the input can be decomposed in a certain way,

then

there is a certain kind of algorithm for it.

Theorem (Courcelle)

lf

- the problem can be described in MSO logic
- and the input has tree width at most k,

then

there is a linear time of algorithm for it.

Theorem (Bodlaender, Courcelle)

lf

- the problem can be described in MSO logic
- and the input has tree width at most k,

then

there is a parallel algorithm running in time O(log n).

Theorem (Courcelle, Makowsky, Rotics)

lf

- the problem can be described in MSO logic
- and the input has clique width at most k,

then

there is a polynomial time algorithm for it.

Theorem (Frick, Grohe)

lf

- the problem can be described in FO logic
- and the input is planar,

then

there is a linear time algorithm for it.

Theorem (Flum, Grohe)

lf

- the problem can be described in FO logic
- and the input has a forbidden minor,

then

there is a polynomial time algorithm for it.

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New Variants 2: Circuit Complexity

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Beyond Time

- While the classical theorems yield tight upper time bounds...
 - ...they yield no completeness results.
- The new metatheorems concern
 - space complexity and
 - circuit complexity
- and yield completeness results.

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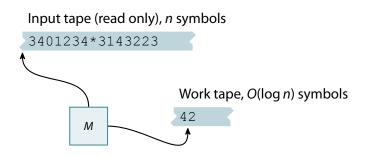
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A Logspace Turing Machine



The Logspace Version of Courcelle's Theorem.

Theorem (Elberfeld, Τ, Jakoby, 2010) Let φ be an MSO formula and k a number. Then

 $\{G \mid G \models \varphi \text{ and } G \text{ has tree width at most } k\}$

can be decided in logspace.

The Low-Hanging Fruits.

Corollary

For all k, we can solve in logspace:

- {G | G is 3-colorable and G has tree width at most k}.
- {G | G has a perfect matching and G has tree width at most k}.
- {(G, s, t) | t is reachable from s and G has tree width at most k}.

(Each result used to be a paper.)



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The High-Hanging Fruits.

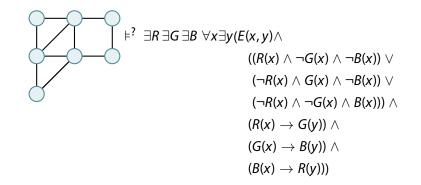
Metatheorems make statements about *decomposable* graphs.

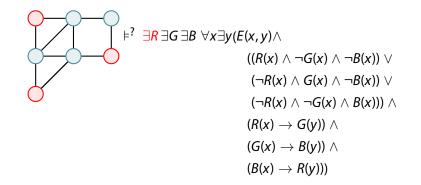
For arbitrary graph, this may work:

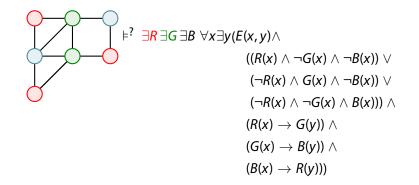
- If the graph is decomposable, apply the metatheorem.
- If the graph is not decomposable, it must have many edges, which we may use algorithmically.

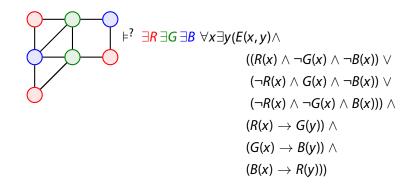


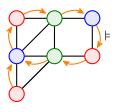
The Formula φ_3 Describes the Existence of Cycles Whose Length is a Multiple of 3.

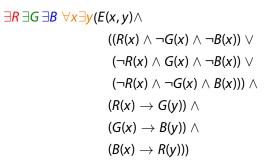


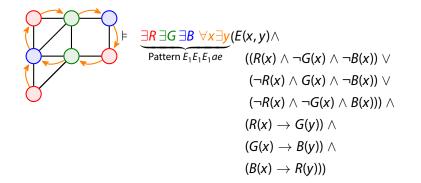












Detecting Cycles Whose Length is a Multiple of 3.

Theorem

There is a logspace algorithm that on input of an undirected graph G decides whether G has a cycle whose length is a multiple of 3.

Proof.

- 1. Check, whether *G* has small tree width.
- 2. If yes, subdivide all edges and apply the metatheorem to φ_4 .
- 3. If no, output "yes".

Thomassen has shown that all graphs of sufficiently large tree width have a cycle whose length is a multiple of 3.

(A similar argument shows EVEN-CYCLE \in L.)

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Formulas in Existential Second-Order Logic Describe NP.

Theorem (Fagin, 1974)

A problem is in NP if, and only if, it can be described by a formula in existential second-order logic. Formulas in Existential Second-Order Logic Describe NP.

Theorem (Fagin, 1974)

A problem is in NP if, and only if, it can be described by a formula in existential second-order logic.

- However, $E_1 E_1 aa$ suffices to describe NP-complete problems.
- What about $E_1 E_1 E_1 ae$?
- What about $E_1 E_1 aa$?
- What about E₁aa?

••••

Can we refine Fagin's Theorem?

Theorem (Gottlob, Kolaitis, Schwentick, 2004)

 E_1^* ae formulas describe problems in P over undirected loop-free graphs.

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 E_1^* ae formulas describe problems in P over undirected loop-free graphs.

Proof.

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a readilish this claim, assume first then $G = (T, X)$ is a graph and X^{μ} is an any relation on T witnessing that G (in $MP_{T}(Y_{T})^{\mu}$, $\mathcal{J}_{T}(x,y)$). Let P^{μ} be				., J.N.Styp, where y is a 2.1, we may many, with,	where $m \in G = G$.	Diam Q		2.40		C is a lost for and.		10 (Same 20, 5)	splacepools	energies data (an. e(B) =	2	
	y vale of a			y) holds. Every antidable interaction of complete con-	a Paralancia of	· v			F	and component K of G.		10.00	in minute		10 75	-
as $G = M^{2}(V_{1}V_{2})_{A_{1}}^{2}$, $F(x, y)$. In the above densities, assume that P^{-1} is a new relation on V with reader that $G = M^{2}(V_{1}V_{2})_{A_{1}}^{2}$, $F(x, y)$. We define the	the state of the second st	1			a substant and its			and the second sec					tion of the nets	W.C.	1.1.2	\rightarrow
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Till Tantau

Theorietag in Speyer, September 2015

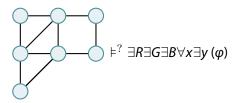
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32/48

Theorem (Gottlob, Kolaitis, Schwentick, 2004)

 E_1^* ae formulas describe problems in P over undirected loop-free graphs.

Proof. Original question:

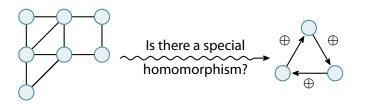


Theorem (Gottlob, Kolaitis, Schwentick, 2004)

 E_1^* ae formulas describe problems in P over undirected loop-free graphs.

Proof.

New question:



Theorem (Gottlob, Kolaitis, Schwentick, 2004)

 E_1^* ae formulas describe problems in P over undirected loop-free graphs.

Proof.

- 1. Preprocess the input graph.
- 2. Check whether the tree width is small.
- 3. If so, we get a polynomial time algorithm by Courcelle's Theorem.
- 4. If not, depending on the target graph,
 - 4.1 we can either just say "yes" or
 - 4.2 test whether a special cycle of *constant length* exists.

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Key Insight

All steps can also be done in *logspace*, in particular also *applying Courcelle's Theorem* because of its logspace version.

Theorem (Gottlob, Kolaitis, Schwentick, 2004)

 E_1^* ae formulas describe problems in P over undirected loop-free graphs.

Theorem (T., 2015)

 E_1^* ae formulas describe problems in L over undirected loop-free graphs.

Corollary EVEN-CYCLE \leftarrow L.

All Pattern Classes Over Undirected Self-Loop Free Graphs

Theorem

The patterns below yield the classes over undirected self-loop free graphs:

$E_{any}^{*}(ae)^{*}$ $E_{1}E_{1}aa$ $E_{1}aaa$ $E_{2}eaa$	E ₁ eae	E ₁ aee	E ₁ aea	E ₁ aae	NP
E ₁ e*aa E ₁ eaa					NL
E _{any} aa E ₁ aa		E, E ₁ E ₁ ae	_{any} ae E ₂ a	е	L
(ae)*	E _{any} e	°a [E ₁ ae		AC ⁰

Outline

Algorithmic Metatheorems

- Classic Variants ...
- ...and New Variants

New Variants 1: Space Complexity

- New Theorems
- Applications: Cycle Lengths in Graphs
- Applications: Quantifier Prefix Classes
- Applications: Integer Optimization

New Variants 2: Circuit Complexity

- New Theorems
- Applications: Visible Pushdown Languages

The Subset Sum Problem

98237534	246301534
3245	2336245
1095345	6245
13435	202934435
39087539	10237
759	59
98798754910	91111074234
- the	

The Subset Sum Problem

Circle some numbers so that they add up exactly to 100000000.

The Subset Sum Problem

98237534	246301534
3245	2336245
1095345	6245
13435	202934435
39087539	10237
759	59
98798754910	91111074234
and the second s	

The Subset Sum Problem

Circle some numbers so that they add up exactly to 1000000000.

This is a well-known NP-complete problem.

The Subset Sum Problem



The Unary Subset Sum Problem

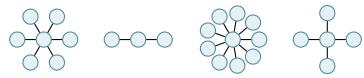
Cirlce some numbers so that they add up exactly to

The Unary Version Is In Logspace.

Theorem UNARY-SUBSET-SUM \in L.

Proof.

For an input like JHT II, III, JHT JHT, JHT consider the forest

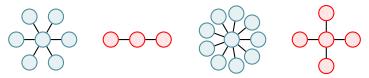


The Unary Version Is In Logspace.

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Proof.

For an input like JHT II, III, JHT JHT, JHT consider the forest



- 1. The size of a set S that is closed under reachability corresponds to a subset sum.
- 2. The formula $\varphi_{closed}(X) = \forall u \forall v[(X(u) \land E(u, v)) \rightarrow X(v)]$ describes "being closed under reachability."
- 3. New goal: Find out whether φ has a solution of a *certain size*.

Theorem (Elberfeld, T, Jakoby, 2010)

Let $\varphi(X)$ be an MSO formula with a free variable and let k be a number. Then there is logspace Turing machine that on input of

- 1. a graph G of tree width at most k and
- 2. a number s

computes the number of subsets S with

- 1. |S| = s and
- 2. *G* ⊧ *φ*(*S*).

What We Are Really Interested In: The *Original* Subset Sum Problem.

Let a_1, \ldots, a_n be the inputs and s the target sum.

Theorem

Dynamic programming solves the subset sum problem in

- time O(ns) and
- space O(s).

For larger *s*, *space* is the bottleneck: 4GB of memory are filled in *one second* for s = 1,000,000,000.



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What We Are Really Interested In: The *Original* Subset Sum Problem.

Let a_1, \ldots, a_n be the inputs and s the target sum.

Theorem

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Theorem (Lokshtanov, Nederlof, 2010)

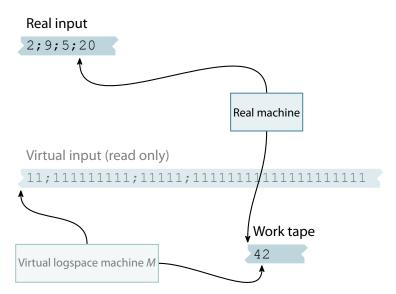
The subset sum problem can be solved in

- time $n^{O(1)}s^{O(1)}$ and
- *space* n^{O(1)}.

Proof.

Complex analysis of approximation circuits for Fourier transformations.

A Magic Trick.



A similar argument shows that there are *space efficient* pseudo-polynomial time algorithms for:

- knapsack problems,
- bin packing problems,
- scheduling problems, and
- integer programming for a fixed number of inequalities.

Outline

Algorithmic Metatheorems

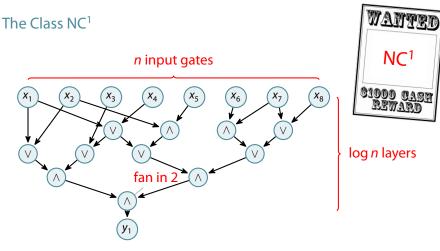
- Classic Variants ...
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New Variants 1: Space Complexity

- New Theorems
- Applications: Cycle Lengths in Graphs
- Applications: Quantifier Prefix Classes
- Applications: Integer Optimization

New Variants 2: Circuit Complexity

- New Theorems
- Applications: Visible Pushdown Languages



- $\blacksquare \operatorname{REG} \subseteq \operatorname{NC}^1 \subseteq \operatorname{L}.$
- Addition, multiplication, and division all lie in NC¹.
- Tree decompositions cannot be computed in NC^1 unless $NC^1 = L$.

Theorem (Elberfeld, T, Jakoby, 2012)

Let φ be an MSO formula and k a number. Then there is an NC¹ circuit family that on input of

1. a graph G

2. together with a tree decomposition of G of width k decides whether $G \models \varphi$.

(As for logspace, there is also a "counting version".)

Outline

Algorithmic Metatheorems

- Classic Variants ...
- ...and New Variants

New Variants 1: Space Complexity

- New Theorems
- Applications: Cycle Lengths in Graphs
- Applications: Quantifier Prefix Classes
- Applications: Integer Optimization

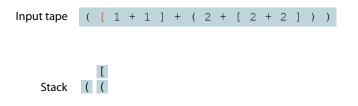
New Variants 2: Circuit Complexity

- New Theorems
- Applications: Visible Pushdown Languages

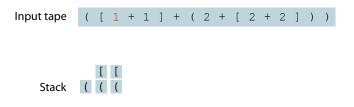
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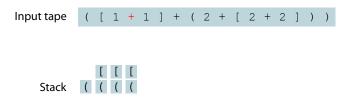
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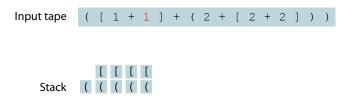
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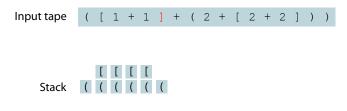
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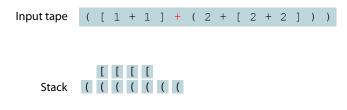
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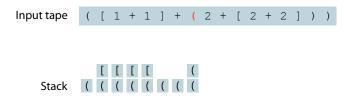
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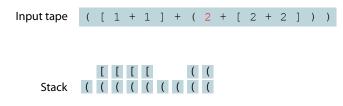
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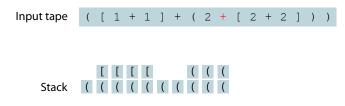
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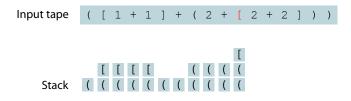
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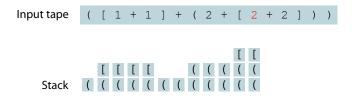
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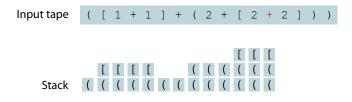
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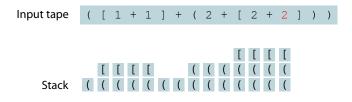
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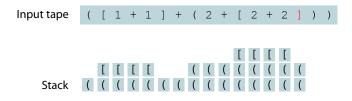
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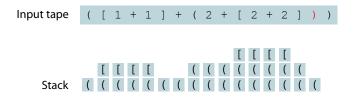
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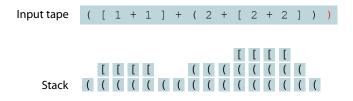
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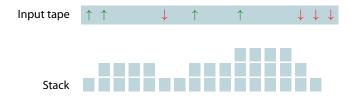


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We Can Precompute the Stack Outline

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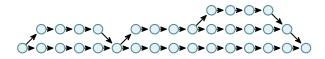


The Complexity of Visible Pushdown Languages

Theorem *All visible pushdown languages lie in* NC¹.

Proof.

For an input like ([1 + 1] + (2 + [2 + 2])) we can construct the following graph using "simple counting":

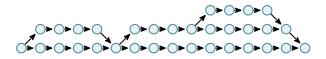


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Apply the metatheorem for NC^1 to a formula stating "there exist states and symbols for the nodes that are locally consistent".

Summary

Algorithmic metatheorems have the form

If a problem can be described in a certain logic and the input graphs can be decomposed in a certain fashion, then there is a certain kind of algorithm.

- There are algorithmic metatheorems for logarithmic space.
- There are algorithmic metatheorems for NC¹.
- Algorithmic metatheorems have applications even when the inputs are not decomposable graphs.

Outlook

Further Results

- There are algorithmic metatheorems for constant depth circuits (AC⁰ and TC⁰).
- There are algorithmic metatheorems for pure logic.

Some Open Problems

- Is there an logspace metatheorem for bounded clique width?
- Can we *construct* solutions of a certain size?
- How difficult is detecting cycles whose length modulo 3 is 1?