Proof. For each of the problems, we reduce its family union version to it. This suffices: By Theorem 3.1 and the fact that the underlying problems like REACH are complete for NL and L under compatible logspace projections (even under first-order projections), the family versions are complete for the respective classes.

Recall that the difference between the problems p-FAMILY-UNION-A and p-SUBSET-UNION-A is that in the first we are given k sets S_i from each of which we must choose one element, while for the latter we can pick k elements from a single set S arbitrarily. If the reduction were to just set S to the union of the S_i, then many choices of k sets of S will correspond to taking multiple elements from a single S_i. In such cases, their union should *not* be an element of A.

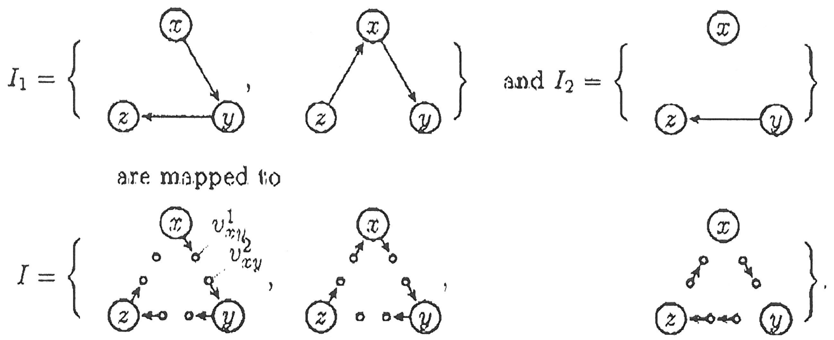

Figure 2: An example of the reduction from a family union graph problem to a subset union graph problem. In the example, $V = \{x, y, z\}$. The s_i are indicated as edge sets even though, in reality, they are bitstrings encoding adjacency matrices. The small vertices are the v^1_{ab} and v^2_{ab}, but only those for $(a, b) \in \{(x, y), (y, z), (z, x)\}$ are shown.

To achieve the effect that the union of a subset of S with multiple elements from the same S_i is not in A, we use the same construction for all A, except for $A =$ FOREST. The construction works as follows: Since the S_i are compatible, they are defined over the same set V of vertices. Each $s \in S_i$ encodes an edge set $E_s \subseteq V^2$. We construct a new vertex set $V' \supseteq V$ as follows: For each pair $(a, b) \in V^2$ we introduce k new vertices $v^1_{ab}, \dots, v^k_{ab}$ and add them to V'. For each $s \in S_i$ we define a new edge set $E'_s \subseteq V' \times V'$ as follows: First, for each $(a, b) \in V^2$ let $(v^{i-1}_{ab}, v^i_{ab}) \in E'_s$, where $v^0_{ab} = a$. Second, for each $(a, b) \in E_s$, let $(v^k_{ab}, b) \in E'_s$. Let s' be the bitstring encoding the adjacency matrix of E'_s. We set $S = \{\, s' \mid s \in S_i \text{ for some } i \}$. An example of how this reduction works is depicted in Figure 2.

In order to argue that the reduction works for all problems, we make two observations. Given any subset $\{s'_1, \dots, s'_k\} \subseteq S$, for each s'_i there is a unique corresponding s_i, lying in (some) S_j. Let $G' = (V', E')$ denote graph whose adjacency matrix is the union of $\{s'_1, \dots, s'_k\}$ and, correspondingly, let $G = (V, E)$ be the union of the S_i. Now, first assume that, indeed, we have $s_i \in S_i$ for all $i \in \{1, \dots, k\}$. Then for every pair $(a, b) \in V$ the new vertices v^1_{ab} to v^k_{ab} will form a path in G' attached to a. Furthermore, for every edge $(a, b) \in E$ there is a path from a to b in E'. On the other hand, for $(a, b) \notin E$, we cannot get from a to b in G' using only new vertices: the edge $v^k_{ab} \to b$ will be missing. This proves our first observation: for vertices $a, b \in V$ there is a path from a to b in G' if, and only if, there is such a path in G. Our second observation concerns the case that there are two strings s'_i and s'_j such that s_i and s_j lie in the same set S_x. In this case, for every two vertices $a, b \in V$ at least one edge is missing along the path v^0_{ab} to v^k_{ab}. Thus, we can observe that there is no path from any $a \in V$ to any other $b \in V$ in G'.