Abstract of the Talk

"Lower Bounds for the Rank of Associative Algebras"

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Let $b_1, \ldots, b_\ell \in k[X_1, \ldots, X_m, Y_1, \ldots, Y_n]$ be bilinear forms in the indeterminates $X_1, \ldots, X_m$ and $Y_1, \ldots, Y_n$ over some field k. The bilinear complexity (also called rank) of this set of bilinear forms is the smallest number r of bilinear products $p_\rho = u_\rho(X) \cdot v_\rho(Y)$, $1 \le \rho \le r$, with linear forms u_ρ and v_ρ in the X's and Y's, respectively, such that each of the $b_1, \ldots, b_\ell$ is contained in the k-linear span of $p_1, \ldots, p_r$. Let A be a finite dimensional associative k-algebra with unity. After fixing some basis, the multiplication map of A induces a set of bilinear forms. The bilinear complexity of these bilinear forms does not depend on the chosen basis. Therefore, we may define the bilinear complexity (also called rank) of A as the bilinear complexity of any of these induced sets of bilinear forms. We denote this quantity by $R(A)$.

In this talk, we are mainly concerned with lower bounds for $R(A)$. The best general lower bound known so far is the so-called Alder–Strassen bound [AS81] $R(A) \ge 2 \dim A - t$, where t is the number of maximal twosided ideals of A. We present the new lower bound $R(A) \ge \frac{5}{2} \dim A - 3(n_1 + \cdots + n_t)$ which holds for any algebra whose decomposition of $A/\operatorname{rad} A \cong A_1 \times \cdots \times A_t$ into simple algebras $A_\tau \cong {D_\tau}^{n_\tau \times n_\tau}$ contains only noncommutative factors, that is, the division algebra D_τ is noncommutative or $n_\tau \ge 2$. In particular, $n \times n$–matrix multiplication requires at least $\frac{5}{2} n^2 - 3n$ essential bilinear multiplications.

A class of algebras that has received a wide attention in this context are those algebras A for which the Alder-Strassen bound is sharp, i.e., $R(A) = 2 \dim A - t$. Such algebras are also called algebras of minimal rank. Building up on the results of [HM86], we characterize the class of all algebras of minimal rank over algebraically closed fields in terms of their algebraic structure.

References

[AS81] A. Alder and V. Strassen. On the algorithmic complexity of associative algebras. *Theoret. Comput. Sci.*, 15:201–211, 1981.

[HM86] J. Heintz and J. Morgenstern. On associative algebras of minimal rank. In *Proceedings of the AAECC 2*, LNCS 228, pages 1–24. Springer, 1986.