The Sublogarithmic Alternating Space World*

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Abstract

This paper tries to fully characterize the properties and relationships of space classes defined by Turing machines that use less than logarithmic space – may they be deterministic, nondeterministic or alternating (DTM, NTM or ATM). We provide several examples of specific languages and show that such machines are unable to accept these languages. The basic proof method is a nontrivial extension of the $1^n \mapsto 1^{n+n!}$ technique to alternating TMs.

Let llog denote the logarithmic function log iterated twice, and $\Sigma_k Space(S)$, $\Pi_k Space(S)$ be the complexity classes defined by S-space-bounded ATMs that alternate at most $k-1$ times and start in an existential, resp. universal state. Our first result shows that for each $k > 1$ the sets

$$\Sigma_k Space(\text{ llog }) \quad \setminus \quad \Pi_k Space(o(\log)) \qquad \text{and}$$
$$\Pi_k Space(\text{ llog }) \quad \setminus \quad \Sigma_k Space(o(\log))$$

are both not empty. This implies that for each $S \in \Omega(\text{ llog }) \cap o(\log)$ the classes

$$\begin{aligned} \Sigma_1 Space(S) \quad &\subset \quad \Sigma_2 Space(S) \subset \Sigma_3 Space(S) \subset \ldots \\ &\subset \quad \Sigma_k Space(S) \subset \Sigma_{k+1} Space(S) \subset \ldots \end{aligned}$$

form an infinite hierarchy. Furthermore, this separation is extended to space classes defined by ATMs with a nonconstant alternation bound A provided that the product $A \cdot S$ grows sublogarithmically.

These lower bounds can also be used to show that basic closure properties do not hold for such classes. We obtain that for any $S \in \Omega(\text{ llog }) \cap o(\log)$ and all $k > 1$ $\Sigma_k Space(S)$ and $\Pi_k Space(S)$ are not closed under complementation and concatenation. Moreover, $\Sigma_k Space(S)$ is not closed under intersection, and $\Pi_k Space(S)$ is not closed under union.

It is also shown that ATMs recognizing bounded languages can always be guaranteed to halt. For the class of Z-bounded languages with $Z \leq \exp S$ we obtain the equality $co\text{-}\Sigma_k Space(S) = \Pi_k Space(S)$.

Finally, for sublogarithmic bounded ATMs we give a separation between the weak and strong space measure, and prove a logarithmic lower space bound for the recognition of nonregular context-free languages.

Key words. space complexity, sublogarithmic complexity bounds, alternating Turing machines, halting computations, complementation of languages, complexity hierachies, closure properties, context-free languages, bounded languages.

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1 Introduction

It is well known that if a deterministic or nondeterministic TM uses less than llog space then the machine can recognize only regular languages, and that there exist non-regular languages in $DSpace(\text{llog})$. Therefore, let $\texttt{SUBLOG} := \Omega(\text{llog}) \cap o(\log)$ denote the set of all nontrivial sublogarithmic space bounds, where llog abbreviates the twice iterated logarithmic function $n \mapsto \lfloor \log\log n \rfloor$. On the other hand, the logarithm seems to be the most dramatic bound for space complexity since most techniques used in space complexity investigations only work for bounds above this threshold. There are several important results for such space classes known, and it is an open question if they also hold for space bounds between llog and $\log$. One of the most exciting problem of this type is whether the closure under complement for NTM

$$NSpace(S) \ = \ co\text{-}NSpace(S)$$

shown by Immerman and Szelépcsenyi [13],[22] remains valid for sublogarithmic space bounds. If this equality were not valid for a function $S \in \texttt{SUBLOG}$ then obviously $DSpace(S) \subset NSpace(S)$ [1].

A special situation holds for bounded languages containing only strings of a certain block structure.

Definition 1 Let $Z : \mathbb{N} \to \mathbb{N}$ be a function. A language $L \subseteq \{0,1\}^*$ is Z*–bounded* if each $X \in L$ contains at most $Z(|X|)$ zeros. L is *bounded* if it is Z–bounded for some constant function Z.

Recently Alt, Geffert, and Mehlhorn ([2]) and independently Szepietowski ([23]) have proved that for the class of Z–bounded languages, where Z is a constant or a small growing function, the closure under complement holds, that means in this case $NSpace(S) = co\text{-}NSpace(S)$ even for sublogarithmic bounds. Still, we conjecture that in general the above result does not hold. Towards this direction we will prove in this paper that $\Sigma_k Space(S)$ is not closed under complementation for any $S \in \texttt{SUBLOG}$ and all $k > 1$.

Recall that for $k \geq 1$ the class $\Sigma_k Space(S)$ is defined as all languages that can be accepted by alternating S space-bounded TMs making at most $k-1$ alternations and starting in an existential state. $\Pi_k Space(S)$ denotes the set of languages accepted by the same kind of machines, except that they start in a universal state. By definition $\Sigma_1 Space(S) = NSpace(S)$. We will also consider ATMs with a non-constant bound A for the number of alternations. In this case, the notation $\Sigma_A Space(S)$ and $\Pi_A Space(S)$ is used.

By standard techniques it follows from Immerman-Szelépcsenyi's theorem that for $S \in \Omega(\log)$, and for all $k \geq 1$

$$\Sigma_1 Space(S) \ = \ \Sigma_k Space(S) \ = \ \Pi_k Space(S) \ .$$

Note that these techniques do not work for sublogarithmic space bounds. Recently, Chang et al. ([7]) have shown that there is a language in $\Pi_2 Space(\text{llog})$ that does not belong to $NSpace(o(\log))$. Clearly, this proves that for space bounds S in $\texttt{SUBLOG}$ the alternating S-space hierarchy does not collapse to the first level and that

$$\Sigma_1 Space(S) \ \subset \ \Pi_2 Space(S) \ .$$

It was left as an open problem whether the whole alternating hierarchy for sublogarithmic space is strict. Here we will prove that the problem has a positive answer.

We develop techniques to investigate properties of sublogarithmic computations and then generalize them to an inductive proof that the separation of the $\Sigma_k Space(S)$ and $\Pi_k Space(S)$ classes holds for all levels k. The base case is the existence of a language that separates $\Pi_2 Space(\text{llog})$ from $\Sigma_2 Space(o(\log))$. Its complement separates $\Sigma_2 Space(\text{llog})$ from $\Pi_2 Space(o(\log))$.

[1] In [25, p.419] it is incorrectly cited that $DSpace(S) \subset NSpace(S)$, for $S \in \texttt{SUBLOG}$, thus the problem if $DSpace(S) = NSpace(S)$ is still open for any $S \in \Omega(\text{llog})$ (see Remark 6.1 in [17]).

Inductively we will construct a sequence of languages $L_{\Sigma k}$ and $L_{\Pi k}$ and prove that $L_{\Sigma k}$ can be recognized by a Σ_k TM with $\text{llog}\, n$ space, but not by any Π_k TM that is $o(\log)$–space bounded. The corresponding claim interchanging Σ_k and Π_k holds for $L_{\Pi k}$. For this purpose, for infinitely many n we will explicitely pinpoint a pair of strings, one string in $L_{\Sigma k}$ and the other one in $L_{\Pi k}$, and show that any sublogarithmic space-bounded Σ_k TM or Π_k TM will make an error on at least one of these strings. Thus we obtain

Theorem 1 For all $k > 1$ holds

$$\begin{array}{lcll} \Sigma_k Space(\text{ llog }) \setminus \Pi_k Space(o(\log)) & \neq & \emptyset & \quad\text{and} \\ \Pi_k Space(\text{ llog }) \setminus \Sigma_k Space(o(\log)) & \neq & \emptyset. & \end{array}$$

This result gives a complete and best possible separation for the sublogarithmic space world, except for the first level $k = 1$. It is left open whether also $\Sigma_1 Space(S) \neq \Pi_1 Space(S)$ for $S \in \texttt{SUBLOG}$. The current techniques do not seem to be applicable to this case.

This separation implies that the sublogarithmic space hierarchy is an infinite one, contrary to the case for logarithmic or larger space bounds.

Corollary 1 For any $S \in \texttt{SUBLOG}$ and all $k \geq 1$ holds

$$\begin{array}{lcl} \Sigma_k Space(S) & \subset & \Sigma_{k+1} Space(S)\ , \\ \Pi_k Space(S) & \subset & \Pi_{k+1} Space(S)\ . \end{array}$$

Independently the existence of this strict hierarchy has been shown by von Braunmühl with coauthors [6]. Geffert [11] has announced similar results. (For a chronology of events see [24].)

Furthermore, we can generalize the separation to machines with an unbounded number of alternations.

Definition 2 A function $A : \mathbb{N} \to \mathbb{N}$ is *computable in space* S if there exists a DTM that for all inputs of the form 1^n writes down the binary representation of $A(n)$ on an extra output tape using no more than $S(n)$ work space. A is *approximable from below in space* S if there exists a function A' that is computable in space S with $A'(n) \leq A(n)$ for all $n \in \mathbb{N}$ and $A'(n) = A(n)$ for infinitely many $n \in \mathbb{N}$.

The class of bounds that are approximable from below in space llog contains functions of logarithmic and double-logarithmic growth and also polynomials of such functions. The iterated logarithm $\log^*$ belongs to this class, too. In section 3 we will discuss a specific example of logarithmic growth.

Theorem 2 For any pair of functions $S \in \texttt{SUBLOG}$ and $A > 1$ with $A \cdot S \in o(\log)$, where A is approximable from below in space S, holds:

$$\begin{array}{lcl} \Sigma_A Space(S) \setminus \Pi_A Space(S) & \neq & \emptyset\ , \\ \Pi_A Space(S) \setminus \Sigma_A Space(S) & \neq & \emptyset\ . \end{array}$$

Corollary 2 For any S and A as in the theorem above holds:

$$\begin{array}{lcl} \Sigma_A Space(S) & \subset & \Sigma_{A+1} Space(S)\ , \\ \Pi_A Space(S) & \subset & \Pi_{A+1} Space(S)\ . \end{array}$$

Thus one obtains for space bounds $S \in \Omega(\text{ llog })$ and approximable functions A for example the following relations:

1. $\bigcup_{k\in\mathbb{N}} \Sigma_k Space(S) \subset \Sigma_{\log^*} Space(S)$ if $S \in o(\frac{\log}{\log^*})$.

2. $\Sigma_A Space(S) \subset \Sigma_{A+1} Space(S)$ for $A, S \in O(\log^{1/2-\epsilon})$,

3. For $k \in \mathbb{N}$ let
$$\mathcal{ALSL}^k := AAlterSpace(\text{llog}^k, \text{ llog}) .$$
Then for any k holds $\mathcal{ALSL}^k \subset \mathcal{ALSL}^{k+1}$.

Note that for logarithmic bounds the corresponding question is still open, i.e. for any k it is unknown whether
$$AAlterSpace(\log^k, \log) \subset AAlterSpace(\log^{k+1}, \log) \ ?$$

It is well known that for any function S the complexity class $\Sigma_1 Space(S)$ is closed under union and intersection (see e.g. [25]). However, it is still an open problem whether for $S \in$ SUBLOG the class $\Sigma_1 Space(S)$ is closed under complementation. More general, for arbitrary k the classes $\Sigma_k Space(S)$ are closed under union, and symmetrically the $\Pi_k Space(S)$ are closed under intersection. In [14] we have developed a technique showing that for $S \in$ SUBLOG and for $k = 2, 3$, $\Sigma_k Space(S)$ and $\Pi_k Space(S)$ are not closed under complementation. Furthermore, $\Sigma_k Space(S)$ is not closed under intersection, and $\Pi_k Space(S)$ not under union. Combining these ideas with the separation results above we get the same closure properties for all levels.

Theorem 3 *For any $S \in$ SUBLOG and all $k > 1$ $\Sigma_k Space(S)$ and $\Pi_k Space(S)$ are not closed under complementation and concatenation. Moreover, $\Sigma_k Space(S)$ is not closed under intersection and $\Pi_k Space(S)$ is not closed under union.*

Note that non-closure under complementation for Σ_k and Π_k classes is not trivially equivalent to Theorem 1, which says that sublogarithmic $\Sigma_k Space$ and $\Pi_k Space$ are distinct. Sublogarithmic space-bounded machines do not have a counter, which could detect an infinite path of computation. It is an interesting open problem whether $\Pi_k Space(S) = co\text{-}\Sigma_k Space(S)$ for $k = 1, 2, \ldots$ (see the discussion in [14]). Here, we obtain the following partial solution generalizing Sipser's result on halting space-bound computation for sublogarithimic space bounded deterministic TMs [19]: For bounded languages it can be shown that there exist equivalent ATMs that always halt. This implies

Theorem 4 *Let $S \in$ SUBLOG be a space bound and Z be a function computable in space S with $Z \leq \exp S$. Then for all $k \geq 1$ and for every Z –bounded language $L \subseteq \{0,1\}^*$ holds:*
$$L \in \Sigma_k Space(S) \iff \overline{L} \in \Pi_k Space(S) .$$

Observe that for $S \geq \log$ the function Z can grow linearly and then Z does not put any restriction on the structure of the strings in L . Thus, this theorem gives a smooth approximation of the fact that for at least logarithmic space bounds Σ_k and Π_k are complementary for arbitrary languages. We conjecture that the computability of Z is needed in the claim above. Furthermore, there are some indications that the theorem might not be true in general for bounds Z much larger than $\exp S$.

Finally, we prove a logarithmic lower space bound for the recognition of context-free languages by ATMs. We will show that the deterministic context-free language $L_{\neq} := \{1^n 0 1^m \mid n \neq m\}$ does not belong to $ASpace(o(\log))$. It is interesting to note that this language – but not its complement – can be recognized even by a deterministic machine in *weak space* llog .

Definition 3 We say that an ATM M is (*strongly*) S space-bounded if on every input X it only enters configurations that use at most $S(|X|)$ space. M is *weakly* S space-bounded if, for every input X that is accepted, it has an accepting computation tree all of which configurations use at most $S(|X|)$ space. $DSpace(S)$ denotes the class of languages accepted by S space-bounded DTMs and $weakDSpace(S)$ denotes the languages accepted by weakly S space-bounded DTMs. A corresponding notation is used for NTMs and ATMs.

In this paper we consider only the more natural strong requirement for space complexity. For at least logarithmic space bounds the two conditions do not make a difference, while in the sublogarithmic case they obviously do. When studying the closure under complement of a language L and alternating hierarchies built on this the weak measure is not appropriate. This is because for strings in $\overline{L}$ a machine for L may use arbitrary much space, while a machine for $\overline{L}$ were required to be bounded. The example above shows that with respect to the weak measure already for DTM $weakDSpace(\text{ llog })$ contains languages that do not belong to $co\text{-}weakDSpace(o(\log))$.

In [7] Chang et al. stated as an open problem whether weak and strong sublogarithmic space-bounded ATMs have the same power. Obviously, our lower space bound for recognizing $L_{\neq}$ by ATMs proves the following

Theorem 5 $weakDSpace(\text{ llog }) \setminus ASpace(o(\log)) \neq \emptyset$.

As consequences one obtains

Corollary 3 *For any* $k \geq 1$ *and each* $S \in \mathtt{SUBLOG}$

$$\begin{array}{rcll} \Sigma_k Space(S) & \subset & weak\Sigma_k Space(S) & \text{and} \\ \Pi_k Space(S) & \subset & weak\Pi_k Space(S). & \end{array}$$

Corollary 4 *For each* $S \in \mathtt{SUBLOG}$

$$ASpace(S) \subset weakASpace(S) .$$

We next generalize the specific lower bound above to arbitrary deterministic context-free languages, which also improves a result for NTMs shown by Alt, Mehlhorn and Geffert [2]. Before stating the result we need the following definition (see [20] and [12]). A language L is called *strictly nonregular* if one can find strings u, v, w, x and y such that $L \cap \{u\}\{v\}^*\{w\}\{x\}^*\{y\}$ is context-free, but nonregular.

Theorem 6 *Let* L *be a nonregular deterministic context-free, a strictly nonregular language, or a nonregular context-free bounded language, then* $L \notin \bigcup_{k \in \mathbb{N}} \Sigma_k Space(o(\log))$. *Furthermore, for ATMs without any bound on the number of alternations it is not possible that* L *and* $\overline{L}$ *both belong to* $ASpace(o(\log))$.

This paper is organised as follows. In the next section the necessary technical tools for sublogarithmic space bounded ATMs will be developed. In section 3 we will define a sequence of pairs of languages indexed by the level number k to prove the sublogarithmic space hierarchy. We then investigate closure properties of sublogarithmic space classes. Section 5 is devoted to the lower space bounds for context-free languages. The paper concludes with a discussion of the most interesting open problems for sublogarithmic space classes remaining.

Preliminary versions of most of these results have been presented in [14] and [15].

2 Properties of Sublogarithmic Space-Bounded ATMs

The Turing machine model we consider is equipped with a two-way read-only input tape and a single read-write work tape. The input word is stored on the input tape between end-markers \$.

Definition 4 A *memory state* of a TM M is an ordered triple $\alpha = (q, u, i)$, where q is a state of M, u a string over the work tape alphabet, and i a position in u (the locaton of the work tape head). A *configuration* of M on an input X is a pair (α, j) consisting of a memory state α and a position j with $0 \leq j \leq |X| + 1$ of the input head. $j = 0$ or $j = |X| + 1$ means that this head scans the left, resp. the right end-marker. For a memory state $\alpha = (q, u, i)$ let $|\alpha|$ denote the length of the memory inscription u.

We may assume that for a successor (α', j') of a configuration (α, j) always holds $|\alpha'| \geq |\alpha|$. The state set of an ATM is partioned into subsets of existential, universal, accepting, and rejecting states. We say that a configuration $((q, u, i), j)$ is existential (resp. universal, accepting, or rejecting) if q has the corresponding mode. All accepting and rejecting configurations C are assumed to be terminating, i.e. there are no more configurations that can be reached from C.

Definition 5 Let

$$(\alpha, i) \quad \models^{\star}_{M,X} \quad (\beta, j)$$

denote the property that the ATM M with X on its input tape has a computation path $C_1 = (\alpha, i), C_2, \ldots, C_t = (\beta, j)$.

$$(\alpha, i) \quad \models_{M,X} \quad (\beta, j)$$

denotes the same fact, but with the following restriction: $t \geq 2$ and the mode of the configurations $C_2, \ldots, C_{t-1}$ is the same as that of C_1 (i.e. if C_1 is existential then all C_l for $l = 2, \ldots, t-1$ are existential, otherwise they are all universal).

$$\mathtt{acc}^k_M(\alpha, i, X)$$

denotes the predicate saying that M starting in configuration (α, i) with X on its input tape accepts (i.e. has an accepting subtree), and on each computation path of that tree it makes at most $k-1$ alternations. Let

$$\mathit{Space}_M(\alpha, i, X)$$

denote the maximum space used in configurations M can reach on input X starting in configuration (α, i) and $\mathit{Space}_M(X) \ := \ \mathit{Space}_M(\alpha_0, 0, X)$, where $(\alpha_0, 0)$ is the initial configuration of M. Similarly let

$$\mathit{Alter}_M(\alpha, i, X)$$

denote the maximum number of alternations M can make on input X starting in configuration (α, i) and $\mathit{Alter}_M(X) \ := \ \mathit{Alter}_M(\alpha_0, 0, X)$.

2.1 Inputs of a Periodic Structure

In this section some properties of TM computations for binary inputs of the form $Z_1 W W \ldots W Z_2$ will be described. Let M be an ATM. Then for any integer $b \geq 0$ we define

$$\mathcal{M}_b \quad := \quad \#\{\alpha \mid \alpha \text{ is a memory state of } M \text{ with } |\alpha| \leq b\} .$$

The following two Lemmata characterize "short" computations i.e. computations restricted to substrings $WW \ldots W$. The first one is a generalization of a result in [16].

Lemma 1 Assume that

$$X \ = \ Z_1 \, W^n \, Z_2$$

where Z_1, W, Z_2 are arbitrary binary strings and $n \in \mathbb{N}$. Moreover let b be an integer and (α, i) and (β, j) configurations with $|\alpha| \leq |\beta| \leq b$ and $|Z_1| < i, j \leq |Z_1 \, W^n|$. Then the following holds:

- If M can go from (α, i) to (β, j) without any alternation and without moving the input head out of the substring W^n then M can also do so such that the head never moves $\mathcal{M}_b^2 \cdot |W|$ or more positions to the left of $\min(i, j)$ nor to the right of $\max(i, j)$.

Proof. We only sketch the main idea. Assume $i \leq j$ and denote by $i_{\min}$ and $j_{\max}$ the furthest position to the left, resp. right of the input head in the computation path of M that starts in (α, i) and ends in (β, j). Let M go from (α, i) to (β, j) moving the input head $\mathcal{M}_b^2 \cdot |W|$ or more positions to the right of j, i.e. $j_{\max} - j \geq \mathcal{M}_b^2 \cdot |W|$. By the pigeon hole principle there exist two positions j_1 and j_2, with $j < j_1 < j_2 < j_{\max}$, and two memory states α' and α'' such that (α', j_1) and (α', j_2) are the last configurations of the computation path from (α, i) to a configuration in which M was at the position $j_{\max}$. Similarly, (α'', j_1) and (α'', j_2) are the first configurations of the computation from the position $j_{\max}$ to (β, j). Then removing the computation paths from (α', j_1) to (α', j_2) and from (α'', j_2) to (α'', j_1) one obtains a computation that starts in (α, i) and ends in (β, j) with the head never moving more than distance $j_{\max} - (j_2 - j_1) < j_{\max}$ to the right of position j. ∎

Lemma 2 Let $|i - j| \geq \mathcal{M}_b^2 \, (\mathcal{M}_b + 1) \cdot |W|$. Assume that M can go from configuration (α, i) to configuration (β, j)

♠ without alternating and without leaving the region between the input positions i and j.

Then,

- there exists an integer $c \in [1..\mathcal{M}_b]$ such that for all $d \in [1..\mathcal{M}_b]$ there is a computation path satisfying (♠) which starts in (α, i) and ends in $(\beta, j - d \cdot \operatorname{sgn}(j - i) \cdot c \cdot |W|)$, where $\operatorname{sgn}(z) := z/|z|$.
- Moreover, there also exists a computation path satisfying (♠) that starts in $(\alpha, i + d \cdot \operatorname{sgn}(j - i) \cdot c \cdot |W|)$ and ends in (β, j).

Proof. In the folowing we will only discuss the case $i < j$ when considering the computation from configuration (α, i) to (β, j). Let $|i - j| \geq \mathcal{M}_b^2(\mathcal{M}_b + 1) \cdot |W|$.

Define for integers $p \geq 0$ the function $h(p) := i + p \cdot |W|$ and let $t := \mathcal{M}_b^2$. Partition integers in $[1 \ldots \mathcal{M}_b^3]$ into the t intervals $[L_1, R_1], [L_2, R_2] \ldots [L_t, R_t]$ of equal length $\mathcal{M}_b$ with boundaries

$$\begin{aligned} L_s &:= (s-1)(\mathcal{M}_b + 1) + 1 \\ R_s &:= L_s + \mathcal{M}_b \, . \end{aligned}$$

For $s \in [1 \ldots t]$ consider all input positions $h(p)$ with $p \in [L_s, R_s]$ and the last configuration of M (before (β, j)) that visits position $h(p)$. Among these $\mathcal{M}_b + 1$ configurations there must exist a pair with positions $p_s < q_s \in [L_s, R_s]$ and identical memory states α_s.

Let $(\gamma_1, i_1) \models_{i,j} (\gamma_2, i_2)$ denote the same property as $(\gamma_1, i_1) \models_{M,X} (\gamma_2, i_2)$, but with the restriction that M going from (γ_1, i_1) to (γ_2, i_2) does not move the head to the left of i nor to the right of j. Then we can write:

$$\begin{array}{llllll} (\alpha, i) & \models_{i,j} & (\alpha_1, h(p_1)) & \models_{i,j} & (\alpha_1, h(q_1)) & \models_{i,j} \\ & & (\alpha_2, h(p_2)) & \models_{i,j} & (\alpha_2, h(q_2)) & \models_{i,j} \\ & & & \cdots & & \\ & & (\alpha_t, h(p_t)) & \models_{i,j} & (\alpha_t, h(q_t)) & \models_{i,j} \quad (\beta, j) \, . \end{array}$$

Since there are t pairs (p_s, q_s) and the difference between any pair is at most $\mathcal{M}_b$, by the pigeon hole principle there exists an integer $c \in [1 \ldots \mathcal{M}_b]$ and $t/\mathcal{M}_b = \mathcal{M}_b$ pairs $(p_{s_1}, q_{s_1}), (p_{s_2}, q_{s_2}), \ldots$ with identical difference c, that means $q_{s_\ell} - p_{s_\ell} = c$ for $\ell = 1, 2, \ldots, \mathcal{M}_b$. Define $\delta' := c \cdot |W|$.

Let d be an arbitrary integer in $[1 \ldots \mathcal{M}_b]$ and define $\alpha'_\ell := \alpha_{s_\ell}$ and $i_\ell := h(p_{s_\ell})$. Then we obtain:

$$\begin{array}{llllll} (\alpha, i) & \models_{i,j} & (\alpha'_1, i_1) & \models_{i,j} & (\alpha'_1, i_1 + \delta') & \models_{i,j} \\ & & (\alpha'_2, i_2) & \models_{i,j} & (\alpha'_2, i_2 + \delta') & \models_{i,j} \\ & & & \cdots & & \\ & & (\alpha'_d, i_d) & \models_{i,j} & (\alpha'_d, i_d + \delta') & \models_{i,j} \quad (\alpha'_{d+1}, i_{d+1}) := (\beta, j) \end{array}$$

The input X contains a sequence of identical blocks W between the positions i and j. For any $\ell \in [1 \dots d]$, M starting in $(\alpha'_\ell, i_\ell + \delta')$ reaches $(\alpha'_{\ell+1}, i_{\ell+1})$ without moving the head to the left of $i_\ell + \delta'$. Therefore M making the same sequence of moves reaches $(\alpha'_{\ell+1}, i_{\ell+1} - \ell\delta')$ when starting in $(\alpha'_\ell, i_\ell + (\ell-1)\delta')$. Thus we obtain

$$\begin{array}{cccc} (\alpha, i) & \models_{i,j} & (\alpha'_1, i_1) & \models_{i,j} \\ & & (\alpha'_2, i_2 - \delta') & \models_{i,j} \\ & & \dots & \\ & & (\alpha'_d, i_d - (d-1)\delta') & \models_{i,j} \quad (\beta, j - d\delta') \ , \end{array}$$

which proves that $(\alpha, i) \models_{i,j} (\beta, j - \delta)$ for $\delta := d \cdot c \cdot |W|$. In a similar way, one can show that there exsists a computation path that starts in $(\alpha, i + \delta)$ and ends in (β, j). ∎

In the following M will always denote an arbitrary ATM and S a space bound in $o(\log)$. Depending on M and S, we choose a constant $\mathcal{N}_{M,S} \geq 2^8$ such that for all $n \geq \mathcal{N}_{M,S}$

$$(\mathcal{M}^6_{S(n)} + 1)^2 \quad < \quad n$$

and

$$S(n) \quad < \quad \frac{1}{2} \log n \ - 2 \ .$$

Remark: In this section all claims following hold for any integer $n \geq \mathcal{N}_{M,S}$.

In [8] Geffert has shown that for sublogarithmic space bounded computations for any natural number ℓ the behavior of a nondeterministic TM on input $1^{n+\ell n!}$ is exactly the same as on 1^n. The proof is based on the so called "$n \to n + n!$ technique" developed by Stearns, Hartmanis, and Lewis in [21]. We will show that a corresponding property holds for ATMs and for all inputs of the form

$$X \ = \ Z_1 \, W^n \, Z_2 \qquad \text{and} \qquad Y \ = \ Z_1 \, W^{n+\ell n!} \, Z_2 \ ,$$

where Z_1, Z_2, W are arbitrary binary strings and $\ell \in \mathbb{N}$.

Since in the following we will often compare computations on such an input X and a pumped version Y let us introduce a special notation for positions within these strings. If i is a position within X outside the pumped region W^n, that means for the example above either in Z_1 or in Z_2, then $\hat{\imath}$ denotes the corresponding position within Y. Thus

$$\hat{\imath} \ := \begin{cases} i & \text{if } i \leq |Z_1| \ , \\ i + |Y| - |X| & \text{if } i > |Z_1 W^n| \ . \end{cases}$$

The main technical tools for the analysis of sublogarithmic space-bounded ATMs are stated in the following Lemmata. Here, X and Y denote strings as defined above and M an arbitrary ATM. Note that n now is not necessarily identical to the length of the input X. Actually, X will in general be much larger than n. But by a repeated application of the following implications we can show that any machine M still obeys a sublogarithmic bound with respect to n.

Lemma 3 (Pumping) Let α, β be memory states with $|\alpha| \leq |\beta| \leq S(n)$, then for any $i, j \in [0 \dots |Z_1|\,] \cup [\,|Z_1 \, W^n| + 1 \dots |X| + 1\,]$ holds:

1. $(\alpha, i) \models_{M,X} (\beta, j) \iff (\alpha, \hat{\imath}) \models_{M,Y} (\beta, \hat{\jmath})$,
2. $(\alpha, i) \models^\star_{M,X} (\beta, j) \iff (\alpha, \hat{\imath}) \models^\star_{M,Y} (\beta, \hat{\jmath})$.

In the analysis below we will use the Pumping Lemma in the following more general form:

Lemma 3′ Let n and m be integers with $\mathcal{N}_{M,S} \leq m \leq (n+1)^2$ and let α, β be memory states with $|\alpha| \leq |\beta| \leq S(m)$. Then for any $i, j \in [0 \dots |Z_1|\,] \cup [\,|Z_1 \, W^n| + 1 \dots |X| + 1\,]$ the properties 1. and 2. above hold.

These claims can be proven using the method developed in [8] and the fact that $\mathcal{M}^6_{S(m)} < n$.

2.2 Space and Alternation Bounds

Lemma 4 (Small Space Bound)

$$Space_M(X) \ \leq\ S(n) \qquad \Longrightarrow \qquad Space_M(Y) \ =\ Space_M(X)\ .$$

Proof. Let $Space_M(X) \leq S(n)$. Assume, to the contrary, that $Space_M(Y) \neq Space_M(X)$. We will show that $Space_M(Y) > Space_M(X)$ cannot occur. A similar contradiction can be obtained for the case $Space_M(Y) < Space_M(X)$.

Assume that $Space_M(Y) > Space_M(X)$. Hence, for Y there exists a computation path $\mathcal{C}$ that starts in the initial configuration $(\alpha_0, 0)$ and ends in a configuration $(\alpha, \hat{\jmath})$ with $|\alpha| = Space_M(X)$ such that from $(\alpha, \hat{\jmath})$ M can reach a configuration $(\beta, \hat{\jmath}')$ with $|\beta| = Space_M(X) + 1$ in one step:

$$(\alpha_0, 0) \ \models^{\star}_{M,Y}\ (\alpha, \hat{\jmath}) \ \models^{\star}_{M,Y}\ (\beta, \hat{\jmath}')\ .$$

If j fulfills the condition $j \leq |Z_1|$ or $j > |Z_1 W^n|$ of the Pumping Lemma then one can conclude immediately:

$$(\alpha_0, 0) \ \models^{\star}_{M,X}\ (\alpha, j) \ \models^{\star}_{M,X}\ (\beta, j')\ .$$

Otherwise, using a similar pumping argument one can show that M on input X can reach a configuration $(\alpha, \bar{\jmath})$, in which the input head is located on W^n and reads the same symbol as in $(\alpha, \hat{\jmath})$. Thus it can also get to memory state β in one more step. We get a contradiction since $|\beta| > Space_M(X)$.

∎

Lemma 5 (Small Alternation Bound)

$$Space_M(X) \ \leq\ S(n) \quad \text{and} \quad Alter_M(X) \ \leq\ \exp S(n) \qquad \Longrightarrow \qquad Alter_M(Y) \ =\ Alter_M(X)\ .$$

Proof. Let i be an integer, with $i \in \{|Z_1|, |Z_1 W^n| + 1\}$ and let α be a memory state, with

$$Space_M(\alpha, i, X) \ \leq\ S(n) \qquad \text{and} \qquad Alter_M(\alpha, i, X) \ \leq\ \exp S(n)\ .$$

Assume that k is an arbitrary positive integer and let

$$\delta_k \ :=\ k \cdot \mathcal{M}_b^2 \cdot (\mathcal{M}_b + 1) \cdot |W|\ ,$$

where $b := S(n)$. We first show that for the input Y the following claim holds:

Claim 1: Let M starting in $(\alpha, \hat{\imath})$ alternate $k-1$ time and never move the input head beyond $W^{n+\ell n!}$. Then there exists a computation of M with $k-1$ alternations that also starts in $(\alpha, \hat{\imath})$, but in which the input head is never moved farther than δ_k positions to the right of $\hat{\imath}$ if $\hat{\imath} = |Z_1|$, resp. to the left of $\hat{\imath}$ if $\hat{\imath} = |Z_1 W^{n+\ell n!}| + 1$.

Proof. We show this claim for $\hat{\imath} = |Z_1|$. The case $\hat{\imath} = |Z_1 W^{n+\ell n!}| + 1$ can be treated similarly. Let us note first that for integers k such that $\delta_k \geq n + \ell n!$ the claim holds trivially. Therefore in the proof below we consider only k with $\delta_k < n + \ell n!$.

Let i' be the smallest integer such that M starting in $(\alpha, \hat{\imath})$ makes $k-1$ alternations with the head never moving to the left of i nor to the right of i'. Assume, to the contrary that $i' > i + \delta_k$. Therefore by the pigeon hole principle there is an interval $[L, R]$, with

$$i + \mathcal{M}_b^2 \cdot |W| \leq L < R \leq i' \quad \text{and} \quad R - L \geq \mathcal{M}_b^2 \cdot (\mathcal{M}_b + 1) \cdot |W|\ ,$$

and a computation path $\mathcal{C}$ of $k-1$ alternations such that M with the head position in $[L, R]$ does not alternate.

Let $\mathcal{C}'$ be a subsequence of configurations of $\mathcal{C}$ of the maximal length such that all configurations of $\mathcal{C}'$ have the head position greater or equal to L and there is a configuration in $\mathcal{C}'$ with the head position i'. Note that the first configuration of $\mathcal{C}'$ equals (α_L, L), for some memory state α_L. Moreover there is a configuration in $\mathcal{C}'$ with the head position R. Let (α_R, R) denote the first such configuration.

Below we show how to cut and paste $\mathcal{C}'$ to obtain a computation path of the same number of alternations but with the head never reaching the position i'. This yields a contradiction to the assumption that $i' > i + \delta_k$.

Let us consider first that $\mathcal{C}'$ is a tail of $\mathcal{C}$. By Lemma 2 there exists a constant c, with $1 \leq c \leq \mathcal{M}_b$ such that M starting in (α_L, L) reaches $(\alpha_R, R - c \cdot |W|)$, with the head positions in $[L, R]$. If additionally M starting in $(\alpha_R, R - c \cdot |W|)$ makes the same sequence of moves as in $\mathcal{C}'$ when started in (α_R, R) then we obtain a computation for M with the same number of alternations as in $\mathcal{C}'$ but with the head never moving to the right of $i' - c \cdot |W|$.

Assume now that $\mathcal{C}'$ is not a tail of $\mathcal{C}$. Then the last configuration of $\mathcal{C}'$ has a form (α'_L, L), for some memory state α'_L. Let (α'_R, R) be the last configuration in $\mathcal{C}'$ with the head position R. By Lemma 2 there exist constants c_1, c_2, with $1 \leq c_1, c_2 \leq \mathcal{M}_b$ such that M starting in (α_L, L) reaches $(\alpha_R, R - c_1 c_2 \cdot |W|)$ and starting in $(\alpha'_R, R - c_1 c_2 \cdot |W|)$ reaches (α'_L, L). It is obvious that M starting in $(\alpha_R, R - c_1 c_2 \cdot |W|)$ and making the same sequence of moves as between (α_R, R) and (α'_R, R) in $\mathcal{C}'$, reaches $(\alpha'_R, R - c_1 c_2 \cdot |W|)$. Hence we obtain a computation path of the same number of alternations as in $\mathcal{C}'$ that starts and ends also in (α_L, L) and (α'_L, L), resp. but with the head never moving to the right of $i' - c_1 c_2 \cdot |W|$. ∎

Note that by Claim 1 and the assumption that $Alter_M(\alpha, i, X) \leq \exp S(n)$ it follows that if M with Y on the input tape starts in $(\alpha, \hat{\imath})$ and makes $k-1$ alternations with the head never moved beyond $W^{n+\ell n!}$ then $k - 1 \leq \exp S(n)$. To see this assume the opposite. Then by Claim 1 M starting in $(\alpha, \hat{\imath})$ makes $k - 1 = \exp S(n) + 1$ alternations such that the head is never moved farther than δ_k positions from $\hat{\imath}$. By the assumption that $n \geq N_{M,S}$ we conclude:

$$\delta_k \;=\; (2^{S(n)} + 2) \cdot \mathcal{M}_b^2 \cdot (\mathcal{M}_b + 1) \cdot |W| \;\leq\; 2^{\frac{1}{2}\log n} \cdot (\mathcal{M}_b^3 + 1) \cdot |W| \;\leq\; n^{1/2} \cdot n^{1/2} \cdot |W| = n \cdot |W| \,,$$

which means that M can make the same computation on X. We obtain a contradiction since $Alter_M(\alpha, i, X) \leq \exp S(n)$. Hence our lemma follows from Claim 1 and from the following

Claim 2: For $k - 1 \leq \exp S(n)$ and for any memory state β and for $j \in \{|Z_1|, |Z_1 W^n| + 1\}$ holds:

M starting in (α, i) with X on the input tape reaches (β, j) with $k-1$ alternations iff
M starting in $(\alpha, \hat{\imath})$ with the input Y reaches $(\beta, \hat{\jmath})$ with $k-1$ alternations.

Proof. We prove the claim for $i = |Z_1|$ and $j = |Z_1 W^n| + 1$. In the other cases a similar proof can be used.

Assume that on input X M reaches (β, j) from (α, i) making $k-1$ alternations. Since

$$k \;\leq\; \exp S(n) + 1 \;\leq\; \lfloor \sqrt{n} \rfloor / 2 \,,$$

there exist non-negative integers n_1, n_2 and n_3 with

$$n_1 + n_2 + n_3 = n \quad \text{and} \quad \lfloor \sqrt{n} \rfloor \leq n_2 \leq n \qquad \text{(i)}$$

such that M alternates only on the prefix $Z_1 W^{n_1}$ and on the suffix $W^{n_3} Z_2$, but not on W^{n_2}. Hence by Lemma 3′, for $i', j' \in [0 \ldots |Z_1 W^{n_1}|] \cup [|Z_1 W^{n_1+n_2}| + 1 \ldots |X| + 1]$, $m' := n$, $n' := n_2$ and $\ell' := \ell n(n-1) \ldots (n_2 + 1)$

$$(\alpha', i') \models_{M,X} (\beta', j') \qquad \Longleftrightarrow \qquad (\alpha', \hat{\imath}') \models_{M,Y} (\beta', \hat{\jmath}')$$

for any configurations (α', i') and (β', j') that are reachable by M on the computation path between (α, i) and (β, j). Using this property one can easily obtain a $(k-1)$–alternating path for input Y that starts in $(\alpha, \hat{\imath})$ and ends in $(\beta, \hat{\jmath})$.

On the other hand if for integers n_1, n_2, n_3 fulfilling (i) there is a computation path for M on Y which starts in $(\alpha, \hat{\imath})$ and ends in $(\beta, \hat{\jmath})$ and such that M does not alternate with the head position in $[|Z_1 W^{n_1}| + 1 \ldots |Z_1 W^{n_1+n_2+\ell' n_2!}|]$ then, applying Lemma $3'$ in the same way as above, one can construct a computation path for the input X which starts and ends in (α, i) and (β, j), resp. and has the same number of alternations. Therefore, to complete the proof we have to show that there exists such a computation path for Y if we assume that M started in $(\alpha, \hat{\imath})$ reaches $(\beta, \hat{\jmath})$ making $k-1$ alternations.

Let m be the largest integer such that for some $n_1, n_3 \in \mathbb{N}$, with $n_1 + m + n_3 = n + \ell n!$, there is a computation path $\mathcal{C}$ between $(\alpha, \hat{\imath})$ and $(\beta, \hat{\jmath})$ of $k-1$ alternations such that M alternates only on the prefix $Z_1 W^{n_1}$ and suffix $W^{n_3} Z_2$. Assume to the contrary that

$$m < \lfloor \sqrt{n} \rfloor + \ell' \lfloor \sqrt{n} \rfloor! \ ,$$

where $\ell' := \ell n(n-1)\ldots(\lfloor \sqrt{n} \rfloor + 1)$. Then either in W^{n_1} or in W^{n_3} there exists a substring of the form $W^{m'}$, with $m' \geq 2\lfloor \sqrt{n} \rfloor$, such that M does not alternate on $W^{m'}$, too. W.l.o.g. let $W^{m'}$ be a substring of W^{n_3}. Then $W^{n_3} = W^{n_3'} W^{m'} W^{n_3''}$ for some integers n_3' and n_3''. Below it is shown that $\mathcal{C}$ can be cut and pasted such that in the new computation path obtained M does not alternate when the input head visits W^{m+1}. This yields a contradiction to the maximality of m.

Let us define the following head position bounds

$$\begin{array}{llll} L_1 & := |Z_1 W^{n_1}| & R_1 & := L_1 + |W^m| + 1 \ , \\ L_2 & := |Z_1 W^{n_1+m+n_3'}| & R_2 & := L_2 + |W^{\lfloor \sqrt{n} \rfloor}| + 1 \ . \end{array}$$

Not that from the assumption that $m' \geq 2\lfloor \sqrt{n} \rfloor$ it follows that

$$R_2 + |W^{\lfloor \sqrt{n} \rfloor}| \leq |Z_1 W^{n+\ell n!}| \ . \tag{ii}$$

Let $\mathcal{C}'$ be a subsequence of computations of $\mathcal{C}$ which starts and ends with the head position in $\{L_1, R_2\}$. We claim that $\mathcal{C}'$ can be modified to the computation path of the same number of alternations, which starts and ends in the same configurations as $\mathcal{C}'$ and such that M does not alternate with the head positions in $[L_1, R_1 + |W|]$. Only the case when $\mathcal{C}'$ starts and ends with the head position L_1 and R_2, resp. will be described.

Let (α_1, L_1) be the first configuration of $\mathcal{C}'$ and (β_2, R_2) the last one. Moreover let (β_1, R_1) be the first configuration in $\mathcal{C}'$ with the head position R_1 and let (α_2, L_2) be the last one with the head position L_2. Using a similar counting argument as in the proof of Lemma 2 one can show that

$$\exists\, c_1 \in [1 \ldots \mathcal{M}_b] \quad \forall\, d \in [1 \ldots \mathcal{M}_b] \qquad (\alpha_1, L_1) \quad \models_{M,Y} \quad (\beta_1, R_1 + c_1 d |W|) \ .$$

Moreover, by Lemma 2 we have:

$$\exists\, c_2 \in [1 \ldots \mathcal{M}_b] \quad \forall\, d \in [1 \ldots \mathcal{M}_b] \qquad (\alpha_2, L_2 + c_2 d |W|) \quad \models_{M,Y} \quad (\beta_2, R_2) \ .$$

Therefore, for $\delta := c_1 c_2 |W|$ holds:

$$\begin{array}{rll} (\alpha_1, L_1) & \models_{M,Y} & (\beta_1, R_1 + \delta) \ , \\ (\alpha_2, L_2 + \delta) & \models_{M,Y} & (\beta_2, R_2) \ . \end{array}$$

By (ii), M making the same moves as in $\mathcal{C}'$ between (β_1, R_1) and (α_2, L_2), reaches $(\alpha_2, L_2 + \delta)$ when started in $(\beta_1, R_1 + \delta)$. Hence, there is a computation path that starts in (α_1, L_1) ends in (β_2, R_2) of the same number of alternations as $\mathcal{C}'$ such that M does not alternate with the head position in $[L_1, R_2 + |W|]$. This completes the proof of the claim and the lemma. ∎

2.3 Fooling ATMs by Pumping the Input

Lemma 6 (1-Alternation) For any configuration (α, i) with

- $i \leq |Z_1|$ or $i > |Z_1 W^n|$ and
- $Space_M(\alpha, i, X) \leq S(n)$ and $Space_M(\alpha, \hat{\imath}, Y) \leq S(n)$ holds:

$$\begin{array}{lcll} \mathtt{acc}^2_M(\alpha, i, X) & \Longrightarrow & \mathtt{acc}^2_M(\alpha, \hat{\imath}, Y) & \text{if } (\alpha, i) \text{ is existential, and} \\ \mathtt{acc}^2_M(\alpha, \hat{\imath}, Y) & \Longrightarrow & \mathtt{acc}^2_M(\alpha, i, X) & \text{for universal } (\alpha, i). \end{array}$$

Proof. Assume that (α, i) fulfils both conditions above. First, let this configuration be existential and let $\mathtt{acc}^2_M(\alpha, i, X)$ be satisfied. Then there exists a universal configuration (or if M does not alternate a final accepting configuration) (β_0, h) with $0 \leq h \leq |X| + 1$, such that

(A) $(\alpha, i) \models_{M,X} (\beta_0, h)$, and

(B) each computation path $\mathcal{C}$ on input X that starts in (β_0, h) is finite. In addition, along each such $\mathcal{C}$ M does not alternate, and the final configuration of $\mathcal{C}$ is accepting.

We divide the string X according to h into three parts. Let $n' := \lfloor n/2 \rfloor$. Define $h_1 := |Z_1 W^{n'}|$ if $h \leq |Z_1 W^{n'}|$, and $h_1 := |Z_1|$, otherwise. Let $h_2 := h_1 + |W^{n'}| + 1$. Now let U denote the prefix of X of length h_1, i.e. $U := Z_1 W^{n'}$ if $h_1 = |Z_1 W^{n'}|$ and $U := Z_1$, otherwise. Moreover let V denote the suffix of X of length $|X| + 1 - h_2$, i.e. if $h_1 = |Z_1 W^{n'}|$ then $V := W^{n-2n'} Z_2$ else $V := W^{n-n'} Z_2$ (note that V can be an empty word). Then, $X = U W^{n'} V$.

For such a partition of X, the head of M in memory state (β_0, h) is located on string $\$U$, if $h \leq |Z_1 W^{n'}|$ and on string $V\$$, otherwise. Let $a := (n'+1)(n'+2)\ldots n$ and let $\ell' := \ell a$. We will show that M started in (α, i) with $X' := U W^{n'+\ell' n'!} V$ on its input tape accepts making at most one alternation. This proves the lemma since

$$X' = U W^{n'+\ell' n'!} V = Z_1 W^{n+\ell' n'!} Z_2 = Z_1 W^{n+\ell n!} Z_2 = Y.$$

Since $\mathcal{N}_{M,S} \leq n \leq (n'+1)^2$ from Lemma $3'$ (for $n := n'$ and $m := n$) and by (A) it follows that

$$(\alpha, \hat{\imath}) \models_{M,X'} (\beta_0, \hat{h})$$

where $\hat{h} := h$ if $h \leq |Z_1 W^{n'}|$ and $\hat{h} := h + \ell' n'!$ otherwise. Our lemma follows from this property and from the fact that

$$\mathtt{acc}^1_M(\beta_0, \hat{h}, X')$$

holds. Below we prove that this predicate is true.

Assume, to the contrary, that $\mathtt{acc}^1_M(\beta_0, \hat{h}, X')$ does not hold. We can distinguish two cases:

(a) $(\beta_0, \hat{h}) \models_{M,X'} (\beta, t)$ for some rejecting or existential configuration (β, t), or

(b) M starting in $(\beta_0, \hat{h})$ performs an infinite universal computation on X'.

From Lemma $3'$, it follows that the memory state β is reachable on X, too. We get a contradiction since by condition (B) it must hold: if M reaches a non-universal memory state on X then it should be accepting. Therefore case (a) cannot occur. Below we will prove that case (b) cannot occur, too. More precisely, we will show that if (b) holds then there exists an infinite universal computation path for input X which starts in $(\beta_0, \hat{h})$, also yielding a contradiction to (B).

Let $\mathcal{C}$ be an infinite universal computation path for input X' that starts in $(\beta_0, \hat{h})$. From $\mathcal{C}$ we will construct an infinite computation path for input X that also starts in $(\beta_0, \hat{h})$. Let $\hat{h}_2$ denote the index of the first symbol of the string $V\$$ on the input tape with input X', i.e. let $hh_2 := h_2 + |W^{\ell' n'!}|$. Three cases have to be distinguished.

Case 1: *The boundary between the prefix U and the string $W^{n'+\ell' n'!}$ or the boundary between the string $W^{n'+\ell' n'!}$ and the suffix V is crossed infinitely often in $\mathcal{C}$ (see the figure below).*

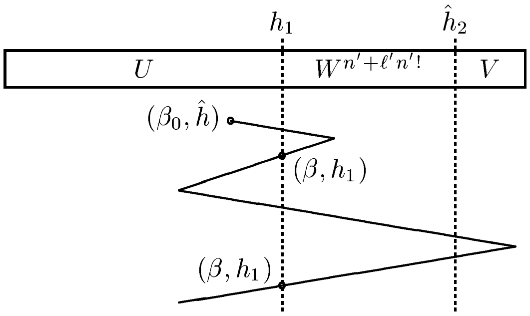


Fig. 1

Let the boundary between the prefix U and the string $W^{n'+\ell' n'!}$ be crossed infinitely many times. Then there exists a memory state β such that the configuration (β, h_1) occurs in $\mathcal{C}$ at least twice. From Lemma $3'$ one can conclude that

$$\begin{array}{lll} (\beta_0, \hat{h}) & \models_{M,X} & (\beta, h_1) \quad \text{and} \\ (\beta, h_1) & \models_{M,X} & (\beta, h_1) \ . \end{array}$$

So, we obtain that M starting in $(\beta_0, \hat{h})$ makes an infinite universal loop on X. The subcase when the boundary between the string $W^{n'+\ell' n'!}$ and the suffix V is crossed infinitely many times in $\mathcal{C}$ is similar to this one.

Case 2: *There is an initial part $\mathcal{C}_1$ of $\mathcal{C}$ and an infinite rest $\mathcal{C}_2$ of $\mathcal{C}$ such that in $\mathcal{C}_2$ M scans only the input to the left of h_1 or to the right of $\hat{h}_2$ (see the figure below).*

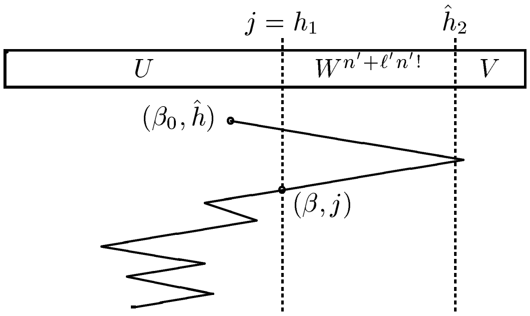


Fig. 2

Let (β, j), for $j = h_1$ or $j = \hat{h}_2$, be the last configuration of $\mathcal{C}_1$. From the Lemma $3'$ we have that (β, j) is reachable from $(\beta_0, \hat{h})$ on X, too. Let $\mathcal{C'}_1$ denote a computation path from $(\beta_0, \hat{h})$ to (β, j) for input X. Then $\mathcal{C'}_1\mathcal{C}_2$ is an infinite computation path for X.

Case 3: *There is an initial part $\mathcal{C}_1$ of $\mathcal{C}$ and an infinite rest $\mathcal{C}_2$ of $\mathcal{C}$ such that in $\mathcal{C}_2$ M scans only the string $W^{n'+\ell' n'!}$ (see the figure below).*

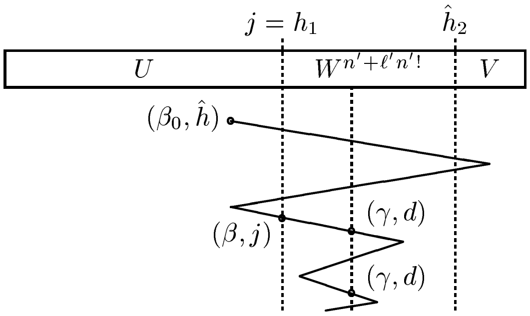


Fig. 3

Let (β, j), for $j = h_1$ or $j = \hat{h}_2$, be the last configuration of $\mathcal{C}_1$. Without loss of generality, assume that $j = h_1$. Since $\mathcal{C}_2$ is infinite there exsists $h_1 < d < \hat{h}_2$ and a memory state γ such that (γ, d) occurs on $\mathcal{C}_2$ at least twice. By assumption, all memory states on computation path between the two instances of (γ, d) use at most $S(n)$ space. Lemma 1 implies that there exists a computation path $\mathcal{D}$ such that $\mathcal{D}$ starts and ends in (γ, d), and that the input head is never moved farther than $\mathcal{M}^2_{S(n)} \cdot |W|$ positions to the left nor to the right of d. Let $\mathcal{C}_2^1$ denote the part of $\mathcal{C}_2$ between (β, j) and the first (γ, d) on $\mathcal{C}_2$. Using Lemma 1 and 2 one can easily construct from $\mathcal{C}_2^1$ a computation path $\mathcal{D}^1$ such that

- $\mathcal{D}^1$ starts in (β, j),
- $\mathcal{D}^1$ ends in (γ, d'), for some d' such that

$$d' < j + \mathcal{M}^2_{S(n)}(\mathcal{M}_{S(n)} + 1) \cdot |W| \quad \text{and} \quad d' \geq \min(d, j + \mathcal{M}^3_{S(n)} \cdot |W|)$$

- the input head is never moved to the left of j nor to the right of

$$j + \mathcal{M}^2_{S(n)}(\mathcal{M}_{S(n)} + 2) \cdot |W| \;\leq\; j + n' \cdot |W| \,.$$

Finally, let $\mathcal{C}_1'$ denote a computation path for input X starting in $(\beta_0, \hat{h})$ and ending in (β, j). By Lemma $3'$ such a path exists. M starting in $(\beta_0, \hat{h})$ and making the same sequence of moves as in $\mathcal{C}_1' \mathcal{D}^1 \mathcal{D}\mathcal{D}\mathcal{D}$... makes an infinite universal loop on X.

This completes the proof of the first implication of the lemma. Let us now assume that $\mathtt{acc}^2_M(\alpha, \hat{\imath}, Y)$ holds for a universal configuration (α, i). If $\mathtt{acc}^2_M(\alpha, i, X)$ is not true then there exists an existential configuration (β_0, h) such that: M starting in (α, i) and working in universal states reaches (β_0, h) and each computation $\mathcal{C}$ of M on X started in (β_0, h) is rejecting or along $\mathcal{C}$ M makes at least one alternation. Using the similar methods as above one can show that $\mathtt{acc}^2_M(\alpha, \hat{\imath}, Y)$ does not hold, too – contradiction. ∎

2.4 Fooling ATMs by Shifting the Input Head

In the following two lemmata we consider the influence of shifting the input head between identical copies of a fixed string W. For this purpose let us denote the shift distance by $\Delta \;:=\; |W| \cdot n!$.

Lemma 7 (Configuration Shift) Let $X \;=\; Z_1\, W^{n+n!}\, W^s\, W^n\, Z_2$ be a binary string with $s \geq 1$ and let α, β be memory states with $|\alpha| \leq |\beta| \leq S(n)$. Then, for any i with $i \leq |Z_1|$ or $i > |Z_1\, W^{n+n!}\, W^s\, W^n|$ and any $j, \ell \in [\, |Z_1\, W^{n+n!}| + 1 \,\ldots\, |Z_1\, W^{n+n!}\, W^s|\,]$ holds:

1. $(\alpha, i) \models_{M,X} (\beta, j) \iff (\alpha, i) \models_{M,X} (\beta, j - \Delta)$,
2. $(\alpha, j) \models_{M,X} (\beta, \ell) \iff (\alpha, j - \Delta) \models_{M,X} (\beta, \ell - \Delta)$,
3. $(\alpha, j) \models_{M,X} (\beta, i) \iff (\alpha, j - \Delta) \models_{M,X} (\beta, i)$.

Proof. First note that the conditions on j and ℓ guarantee that all positions j, ℓ, $j - \Delta$, $\ell - \Delta$ considered are at least n blocks W away from the boundaries Z_1 and Z_2. Define

$$X' \;:=\; Z_1\, W^n\, W^s\, W^n\, Z_2 \quad \text{and} \quad X'' \;:=\; Z_1\, W^n\, W^s\, W^{n+n!}\, Z_2 \,.$$

Set $\hat{\imath} := i$ if $i \leq |Z_1|$, otherwise $\hat{\imath} := i - \Delta$. Using the Pumping Lemma twice – first for the input pair X, X' and then for X', X'' – we obtain:

$$\begin{array}{ccccc}
(\alpha, i) \models_{M,X} (\beta, j) & \iff & (\alpha, \hat{\imath}) \models_{M,X'} (\beta, j - \Delta) & \iff & (\alpha, i) \models_{M,X''} (\beta, j - \Delta) \\
(\alpha, j) \models_{M,X} (\beta, \ell) & \iff & (\alpha, j - \Delta) \models_{M,X'} (\beta, \ell - \Delta) & \iff & (\alpha, j - \Delta) \models_{M,X''} (\beta, \ell - \Delta) \\
(\alpha, j) \models_{M,X} (\beta, i) & \iff & (\alpha, j - \Delta) \models_{M,X'} (\beta, \hat{\imath}) & \iff & (\alpha, j - \Delta) \models_{M,X''} (\beta, i)
\end{array}$$

The claim of the lemma follows because $X'' = X$. ∎

In the inductive argument for the proof of Theorem 1 (Proposition 1 in section 3 below) we have to guarantee a certain distance of the input head from the boundaries. For this purpose we define

$$m_{k,n} \;:=\; k \cdot (n + n!)\ .$$

Lemma 8 (Position Shift) Let $k \geq 2$, r, s, t be integers with $r, t \geq m_{k,n}$ and $s \geq 1$, and let $Z_1, Z_2, W \in \{0,1\}^*$ be arbitrary strings. Then for $X = Z_1\, W^r\, W^s\, W^t\, Z_2$ and for any configuration (α, i) fulfilling the requirements

1. $|Z_1\, W^r| \;<\; i \;\leq\; |Z_1\, W^r\, W^s|$ and
2. $Space_M(\alpha, i, X) \;\leq\; S(n)$ and $Space_M(\alpha, i - \Delta, X) \;\leq\; S(n)$

holds:

$$\mathtt{acc}_M^{k-1}(\alpha, i, X) \qquad \Longleftrightarrow \qquad \mathtt{acc}_M^{k-1}(\alpha, i - \Delta, X)\ .$$

Proof. Let input X and configuration (α, i) be as above. We will only give a proof for

$$\mathtt{acc}_M^{k-1}(\alpha, i, X) \qquad \Longrightarrow \qquad \mathtt{acc}_M^{k-1}(\alpha, i - \Delta, X)\ .$$

A similar argument yields the opposite implication. Let

$$\mathtt{acc}_M^{k-1}(\alpha, i, X) \tag{i}$$

be true. First we will show the following property for computations that start in $(\alpha, i - \Delta)$. Call a computation path of finite or infinite length *universal* if all its configurations are universal.

Claim 1 For a universal configuration (α, i) of M on X any universal computation path that starts in $(\alpha, i - \Delta)$ is finite.

Proof. Let us assume, to the contrary, that there exists an infinite universal computation path that starts in $(\alpha, i - \Delta)$. Hence there exists a universal configuration (β, j) such that

$$(\alpha, i - \Delta) \models_{M,X} (\beta, j) \quad \text{and} \quad (\beta, j) \models_{M,X} (\beta, j)\ .$$

If $|Z_1 W^n| < j \leq |Z_1 W^{r+s+t-(n+n!)}|$ then Lemma 7 (2.) implies

$$(\alpha, i) \models_{M,X} (\beta, j + \Delta) \quad \text{and} \quad (\beta, j + \Delta) \models_{M,X} (\beta, j + \Delta)\ .$$

This means that in (α, i) M starts an infinite universal computation path with X on its input tape. This yields a contradiction to $\mathtt{acc}_M^{k-1}(\alpha, i, X)$.

On the other hand, if $j \leq |Z_1 W^n|$ or $j > |Z_1 W^{r+s+t-(n+n!)}|$ then by Lemma 7 (3.)

$$(\alpha, i) \models_{M,X} (\beta, j)\ .$$

Since $(\beta, j) \models_{M,X} (\beta, j)$ M also generates an infinite universal computation from (α, i). Note that we can apply the Configuration-Shift-Lemma both to α and β because by the second assumption $|\alpha| \leq |\beta| \leq S(n)$. This ends the proof of Claim 1. ∎

First we will solve the base case $k = 2$ and consider an existential configuration (α, i). Because of $\mathtt{acc}_M^1(\alpha, i, X)$ there exists an accepting (β, j) with

$$(\alpha, i) \models_{M,X} (\beta, j)\ .$$

Using the Configuration-Shift-Lemma one can conclude that

$$\begin{array}{llll} (\alpha, i-\Delta) & \models_{M,X} & (\beta, j-\Delta) & \text{if } |Z_1 W^{n+n!}| < j \le |Z_1 W^{r+s+t-n}|, \text{ and} \\ (\alpha, i-\Delta) & \models_{M,X} & (\beta, j) & \text{otherwise.} \end{array}$$

Since β is accepting $\mathtt{acc}^1_M(\alpha, i-\Delta, X)$ holds.

For universal configurations (α, i) it will be shown that any terminating configuration (β, j) with $(\alpha, i-\Delta) \models_{M,X} (\beta, j)$ is accepting. Together with Claim 1 this proves that $\mathtt{acc}^1_M(\alpha, i-\Delta, X)$ holds. Let (β, j) with $(\alpha, i-\Delta) \models_{M,X} (\beta, j)$ be a final configuration. By Lemma 7

$$(\alpha, i) \models_{M,X} (\beta, j+\Delta)$$

if $|Z_1 W^n| < j \le |Z_1 W^{r+s+t-(n+n!)}|$, otherwise

$$(\alpha, i) \models_{M,X} (\beta, j) .$$

Hence, if β is non-accepting then $\mathtt{acc}^1_M(\alpha, i, X)$ does not hold – a contradiction.

Now let $k > 2$ and consider an existential configurations (α, i). Since, by assumptions, M starting in (α, i) with X on the input tape accepts there exists an existential computation path ending in a universal configuration (β, j), with

$$(\alpha, i) \models_{M,X} (\beta, j) , \tag{ii}$$

and

$$\mathtt{acc}^{k-2}_M(\beta, j, X) . \tag{iii}$$

(The trivial case that M accepts without alternations could be handled as above.) Let us divide the input $X = Z_1\, W^r\, W^s\, W^t\, Z_2$ into three regions A, B, C as follows:

$$\begin{array}{lll} A & := & Z_1 W^{r-(n+n!)} , \\ B & := & W^{n!} W^n W^s W^n , \\ C & := & W^{t-n} Z_2 . \end{array}$$

According to j, the input head position in configuration (β, j), the following situations will be distinguished:

Case 1. The input head is located in region A or C (see Fig. 4a), i.e. $j \le |A|$ or $j > |AB|$.

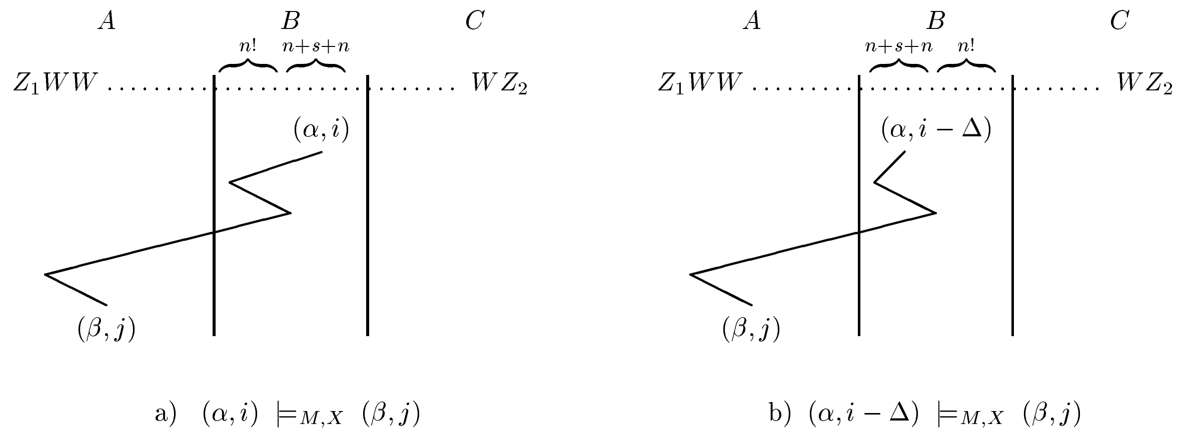


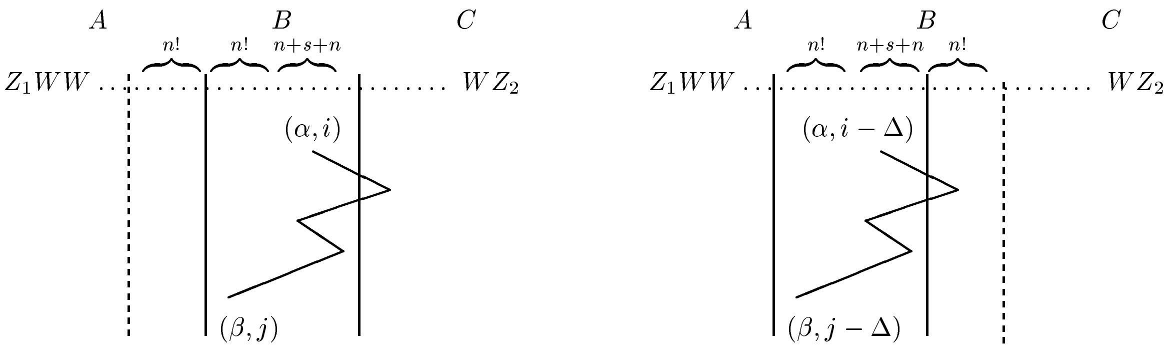
Fig. 4

From property (ii) and Lemma 7 (3.) – for $Z_1 := A$, and $Z_2 := C$ – we obtain that

$$(\alpha, i - \Delta) \models_{M,X} (\beta, j)$$

(see Fig. 4b). Therefore condition (iii) implies $\mathtt{acc}_M^{k-1}(\alpha, i - \Delta, X)$.

Case 2. The input head in (β, j) visits region B (see Fig. 5), i.e. $|A| < j \leq |AB|$.



a) $(\alpha, i) \models_{M,X} (\beta, j)$ b) $(\alpha, i - \Delta) \models_{M,X} (\beta, j - \Delta)$

Fig. 5

In this case using property (ii) and Lemma 7 (2.) – for $Z_1' := Z_1\, W^{r-2(n+n!)}, Z_2' := W^{t-2n} Z_2$, and $s' := n! + n + s + n$ – we conclude that

$$(\alpha, i - \Delta) \models_{M,X} (\beta, j - \Delta) .$$

Now apply the induction hypothesis for $k-1$ with parameters $r' := r - (n + n!)$, s' and $t' := t - n$ to configuration (β, j). By definition of the parameters $m_{k,n}$ the requirements 1. and 2. are fulfilled. Therefore (iii) implies

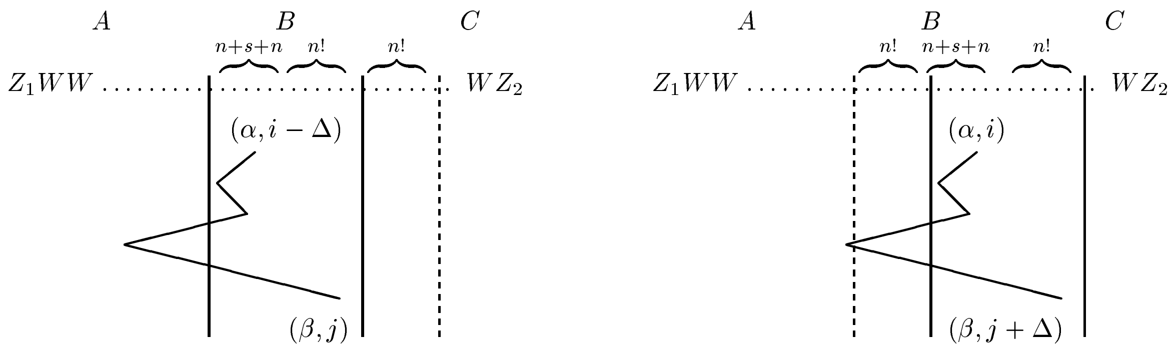
$$\mathtt{acc}_M^{k-2}(\beta, j - \Delta, X) ,$$

and hence $\mathtt{acc}_M^{k-1}(\alpha, i - \Delta, X)$. This completes the proof for existential configurations.

For a universal (α, i), similar to the case $k = 2$, it will be shown that for any final or existential configuration (β, j) that ends a universal computation path

$$(\alpha, i - \Delta) \models_{M,X} (\beta, j) \quad \text{implies} \quad \mathtt{acc}_M^{k-2}(\beta, j, X) .$$

Remember that because of Claim 1 only finite paths have to be considered. Let (β, j) be such a configuration. Divide the input X into three regions A, B, C as above. Depending on which region is visited by the input head in configuration (β, j), two cases are considered. If the input head is in region A or C (as in Fig. 4b) then from Lemma 7 (3.) we obtain that $(\alpha, i) \models_{M,X} (\beta, j)$. $\mathtt{acc}_M^{k-1}(\alpha, i, X)$ thus implies $\mathtt{acc}_M^{k-2}(\beta, j, X)$.



a) $(\alpha, i-\Delta) \models_{M,X} (\beta, j)$ b) $(\alpha, i) \models_{M,X} (\beta, j+\Delta)$

Fig. 6

Otherwise the input head is located in B, i.e. $|A| < j \leq |AB|$ (see Fig. 6a). By Lemma 7 (2.), one can deduce that $(\alpha, i) \models_{M,X} (\beta, j+\Delta)$, which implies $\mathtt{acc}_M^{k-2}(\beta, j+\Delta, X)$. Using the induction hypothesis for configuration $(\beta, j+\Delta)$ and for $k-1$ with $r' := r-n$, $s' := n+s+n+n!$ and $t' := t-(n+n!)$ we obtain $\mathtt{acc}_M^{k-2}(\beta, j, X)$, which completes the proof. ∎

2.5 Halting Computations for ATMs

Let S and Z be functions such that Z is computable in space S and $Z \leq \exp S$. We say that a binary string X is Z*-bounded* if it contains at most $Z(|X|)$ zeros.

Lemma 9 *For every S-space-bounded ATM M there exists an ATM M', which is also S-space-bounded, such that for all Z-bounded strings X holds:*

- *M' accepts X iff M accepts X,*
- *$Alter_{M'}(X) \leq Alter_M(X)$,*
- *if $Alter_M(X) < \infty$ then every computation path of M' on X is finite.*

Proof. Let M be an ATM and let X be a Z-bounded input. In the proof below, $\mathcal{M}_b$ denotes the number of memory states of M as defined in Section 2.1.

Let a *crossing* be any transition of M from a configuration, in which it reads an input symbol a to a configuration reading an input symbol $b \neq a$, where $a, b \in \{0,1\} \cup \{\$\}$. A sequence $\mathcal{C} = C_u, C_{u+1}, \ldots, C_v$ of consecutive configurations of a computation path on X is a *long turn* if $\mathcal{C}$ does not contain alternations, nor crossings, if in C_u and C_v the input head is at the same position i for some $1 \leq i \leq |X|$, and within $\mathcal{C}$

– either the input head visits position $i + \mathcal{M}_b^2$, but never moves to the left of i,

– or it visits position $i - \mathcal{M}_b^2$, but never moves to the right of i,

where b is the amount of space used in C_v.

On the other hand, a sequence $\mathcal{C}$ without alternations or crossings is a *long hop* if the positions i and j of the input head in C_u, resp. C_v are at least at a distance $\mathcal{M}_b^2 + 1$ apart and within $\mathcal{C}$ the input head never leaves the region between these two positions.

Now we are ready to describe the behaviour of the machine M'. It first computes the value $Z(|X|)$, which by assumption can be done in space $S(|X|)$, and then simulates M step by step. Let b_t be the amount of work space used by M by its t-th step.

After having simulated step t of M the machine M' stops and *rejects* iff

a1) M rejects at this step, or

a2) M has just finished a long turn that contains only existential configurations, or

a3) since its last alternation M has executed $2(Z(|X|)+1)\cdot\mathcal{M}_{b_t}+1$ many crossings, or

a4) within the last $2\mathcal{M}_{b_t}^3+1$ steps M has not made any *progress*, that means performed an alternation, a crossing, a long turn or a long hop.

M' stops and *accepts* iff

b1) M accepts, or

b2) M has just finished a long turn that contains only universal configurations.

To check these conditions one counter for the number of crossings, one counter for the number of steps since the last progress and a sliding window for the most recent furtherst distance to the right or left, which can also be realized by counters, suffice. The length of all counters is bounded by $O(S(|X|))$. Thus, M' is $O(S)$-space bounded.

It is obvious that $Alter_{M'}(X)\leq Alter_M(X)$. To see that all computations of M' are finite, first notice that if M does not make progress inifinitely often M' will stop the simulation eventually. Assume that M' does not stop on some path. If $Alter_M(X)<\infty$ this cannot be due to alternations nor to crossings of M since there is also a finite bound set by M'. Thus it remains the case that M within one block of identical input symbols performs infinitely many steps without an alternation. M' would stop if M makes a long turn, thus M has to make an unbounded number of long hops. After a long hop to one side it cannot make a long hop to the other side, because this would result in a long turn. Thus, M eventually has to reach the boundary of this block and performs a crossing, a contradiction.

From Lemma 1 follows that M' accepts the same set of Z-bounded strings as M. In case a2) there is a shorter turn that brings M into a configuration identical to C_v. Thus, if M has an accepting subtree for configuration C_u then it still has after chopping of that C_v which is reached by the long turn. The dual argument holds in case b2). Observe that in case a3) M must have gone through a loop and one can stop the simulation. This is because there are at most $2(Z(|X|)+1)$ different positions on the input tape (counting both directions) to perform a crossing on a Z-bounded string X. Hence, at some position a memory state must repeat. A similar argument holds in case a4) for the at most $\mathcal{M}_b^2$ many input positions that can be visited without performing a long turn or hop. ∎

Using this lemma we can show the following theorem that extends Sipser's space-bounded halting result to alternating TMs.

Theorem 7 *Let S,A,Z be bounds with $A<\infty$ and $Z\leq\exp S$ computable in space S. Then for every S-space-bounded Σ_A TM M there exists a Σ_A TM M' of space complexity S such that for all inputs X*

- *M' accepts X iff M accepts X and X is Z-bounded, and*
- *every computation path of M' on X is finite.*

The identity of Σ_k and co-Π_k for Z-bounded languages (Theorem 4) now follows easily.

3 Hierarchies

3.1 Technical Preliminaries

As a specific example of a function that can be computed in sublogarithmic space consider the following function from [3]

$$F(n) \ := \ \min\{k \in \mathbb{N} \mid k \text{ does not divide } n\} \ .$$

It is easy to see that $F \in O(\log)$. Thus, on input 1^n a TM can simply try all candidate $k = 2, 3, \ldots$ by counting the input length $\bmod k$ until the first nondivisor is found. Using the binary representation this requires at most $\log F(n) \ \leq \ \text{llog}\, n + O(1)$ space.

Obviously, F takes constant values like 2 or 3 infinitely often. We want to show that also the logarithmic upper bound is achieved infinitely often. This would imply that there exists another function G of logarithmic growth that can be approximated from below in space llog . Let $p_1 < p_2 < \ldots$ be the standard enumeration of primes and define

$$\begin{aligned}
\Phi(k) \ &:= \ \prod_{p_i \leq k} p_i^{\lfloor \log_{p_i} k \rfloor} \ , \\
\Phi^{-1}(n) \ &:= \ \min\{k \mid \Phi(k) \geq n\} \ , \\
G(n) \ &:= \ \min\{\ell \mid \ell > \Phi^{-1}(n) \text{ and } \ell \text{ is a prime power } \} \ .
\end{aligned}$$

The following properties can easily be derived.

1. $\Phi^{-1}(\Phi(k)) = k$ and $\Phi(\Phi^{-1}(n)) \geq n$.
2. $F(\Phi(k)) = G(\Phi(k))$, since any $\ell \leq k$ divides $\Phi(k)$ and the first nondivisor in the sequence $k+1, k+2, \ldots$ must be a prime power.
3. $F(n) \leq G(n)$ for all n , which can be seen as follows: Let $k = \Phi^{-1}(n)$. Since we have already considered the case $n = \Phi(k)$ due to property 1. we may assume $n < \Phi(k)$. By definition of Φ there must exist a prime power $p_i^{\lfloor \log_{p_i} k \rfloor}$ that is not a divisor of n . Thus, $F(n) \leq k < G(n)$.
4. $\Phi(k) \ = \ e^{k(1+o(1))}$:
   The prime number theorem implies
   $$\prod_{p_i \leq k} p_i \ = \ e^{k(1+o(1))} \ .$$
   Thus, $\Phi(k) \geq e^{k(1+o(1))}$. On the other hand,
   $$\Phi(k) \ \leq \ \prod_{p_i \leq \sqrt{k}} p_i^{\log_{p_i} k} \ \cdot \prod_{\sqrt{k} < p_i \leq k} p_i \ \leq \ \prod_{p_i \leq \sqrt{k}} k \ \cdot \ \prod_{p_i \leq k} p_i \ \leq \ e^{(\sqrt{k}\, \ln k + k)(1+o(1))} \ = \ e^{k(1+o(1))} \ .$$
5. $\Phi^{-1}(n) \ = \ \ln n \ (1 + o(1))$.
6. $G(n) \ = \ \Phi^{-1}(n)(1+o(1)) \ = \ \ln n \ (1+o(1))$, since any interval $[\Phi^{-1}(n), \Phi^{-1}(n) \cdot (1+o(1))]$ is guaranteed to contain a prime.

Hence, the function G is of logarithmic growth and approximated from below by F .

Let $\mathcal{F}$ be an infinite subset of the natural numbers with the following property:

$$(\spadesuit) \qquad n \in \mathcal{F} \qquad \Longrightarrow \qquad n + n! \notin \mathcal{F} \ .$$

Using the function F we can give a simple example for such a set $\mathcal{F}$ (compare [7]):

$$\mathcal{F} \ := \ \{n > 2 \mid \forall\, \ell \in [3 \ \ldots \ n-1] \quad F(\ell) < F(n)\} \ .$$

The following property of $\mathcal{F}$ will be needed in the lower bound proofs.

Lemma 10
1.) Every interval of the form $[m, m^3]$ with $m \geq 3$ contains an element of $\mathcal{F}$.
2.) For any integer $n > 2$ holds $n + n! \notin \mathcal{F}$.

Proof. Since the function F is not bounded the set $\mathcal{F}$ is infinite. More specific, $\mathcal{F}$ contains all numbers of the form $\Phi(p_k)$ because $F(\Phi(p_k)) > p_k$ and for all $n < \Phi(p_k)$ by the same argument as in 3. above $F(\Phi(p_k))) \leq p_k$. The first claim can be shown by estimating the density of the sequence $(\Phi(p_k))_{k=1,2,\ldots}$. Since $p_{k+1} \leq 2p_k$ for all k we get

$$\Phi(p_{k+1}) \;=\; \prod_{p_i \leq p_{k+1}} p_i^{\lfloor \log_{p_i} p_{k+1} \rfloor} \;\leq\; \prod_{p_i \leq p_{k+1}} p_i^{1+\lfloor \log_{p_i} p_k \rfloor} \;=\; \Phi(p_k) \cdot \prod_{p_i \leq p_{k+1}} p_i \;\leq\; \Phi(p_k)^3 \;.$$

2.) follows easily from the equation $F(n) = F(n+n!)$. To see this equality note that any divisor of n divides $n+n!$, too. Hence $F(n) \leq F(n+n!)$. On the other hand from the definition of F we know that

$$F(n) \text{ does not divide } n$$

and, since $F(n) \leq n$, that

$$F(n) \text{ divides } n! \;.$$

Therefore $F(n)$ does not divide $n+n!$, which means that $F(n+n!) \leq F(n)$. ∎

3.2 ATMs with a Constant Number of Alternations

With the help of sets $\mathcal{F}$ as defined above we construct a sequence of languages that separate the different levels of the alternation hierarchy for sublogarithmic space-bounded ATMs.

Definition 6 For an infinite subset $\mathcal{F}$ of the natural numbers let $L_{\mathcal{F}}$ be the language over the single letter alphabet $\{1\}$ given by $1^n \in L_{\mathcal{F}}$ iff $n \in \mathcal{F}$.
Assume that $\mathcal{F}$ has property ($\spadesuit$) and that $L_{\mathcal{F}} \in \Pi_2 Space(\text{llog})$ and $\overline{L}_{\mathcal{F}} \in \Sigma_2 Space(\text{llog})$.
Then we define $L_2 := \{1\}^+$, and for $k \geq 3$ $L_k := (L_{k-1}\ \{0\})^+$. Furthermore,

$$\begin{aligned}
L_{\Pi 2} &:= L_{\mathcal{F}} \quad \text{and} \quad L_{\Sigma 2} := \{1\}^+ \cap \overline{L}_{\mathcal{F}} \;, \\
L_{\Sigma k} &:= \{w_1 0 w_2 0 \ldots 0 w_p 0 \mid p \in \mathbb{N},\ w_i \in L_{k-1} \text{ and } \exists\, i \in [1...p] \quad w_i \in L_{\Pi k-1}\} \;, \\
L_{\Pi k} &:= \{w_1 0 w_2 0 \ldots 0 w_p 0 \mid p \in \mathbb{N},\ w_i \in L_{k-1} \text{ and } \forall\, i \in [1...p] \quad w_i \in L_{\Sigma k-1}\} \;.
\end{aligned}$$

Note that $L_{\Sigma 2}$ and $L_{\Pi 2}$ are just complementary. For larger k the corresponding languages are "almost" complementary, that means if restricting to strings with a syntactically correct division into subwords by the 0-blocks (more formally $L_{\Pi k} = L_k \cap \overline{L}_{\Sigma k}$).

Lemma 11 For the specific $\mathcal{F}$ defined above with the help of the function F holds

$$L_{\mathcal{F}} \in \Pi_2 Space(\text{llog}) \quad \text{and} \quad \overline{L}_{\mathcal{F}} \in \Sigma_2 Space(\text{llog}) \;.$$

Proof. We describe llog space-bounded Π_2TMs M_Π and Σ_2TMs M_Σ that recognize the language $L_{\mathcal{F}}$, resp. the complement of $L_{\mathcal{F}}$. The machine M_Π verifies the condition $\forall\, \ell \in [3\ \ldots\ n-1] \quad F(\ell) < F(n)$ as follows:

- deterministically it computes $F(n)$ and writes down the binary representation of $F(n)$ on the tape;
- universally it guesses an integer $\ell \in [3 \ldots n-1]$:
  it moves its input head to the right and stops ℓ positions from the right end of the string 1^n;

– existentially it guesses an integer $k \in [1 \dots F(n)-1]$ and then moving the input head to the right, checks deterministically whether k divides ℓ.
  M_Π accepts if k does not divide ℓ.

The complementary machine M_Σ writes down on the work tape $F(n)$ in binary and tests whether

$$\exists\, \ell \in [3 \;\dots\; n-1] \quad \forall\, k \in [1 \;\dots\; F(n)-1] \qquad k \text{ divides } \ell .$$

Similarly as in M_Π the input head position represents the integer ℓ. The integer k is stored in binary on the work tape. It is obvious that M_Π recognizes $L_\mathcal{F}$ and that M_Σ recognizes $\overline{L}_\mathcal{F}$ in space $O(\text{llog})$. ∎

Thus languages $L_\mathcal{F}$ as assumed in Definition 6 exist. For the base case of the following inductive separation we also need the property that $L_\mathcal{F} \notin \Sigma_2 Space(o(\log))$ and symmetrically that $\overline{L}_\mathcal{F} \notin \Pi_2 Space(o(\log))$. This has been shown for the example above explicitely in [14]. Below we will give a general argument showing that this property simply follows from the condition $n \in \mathcal{F}$ and $n + n! \notin \mathcal{F}$.

Lemma 12 For any $k \geq 2$ holds

$$\begin{array}{rcl} L_{\Sigma k} & \in & \Sigma_k Space(\text{ llog }) , \\ L_{\Pi k} & \in & \Pi_k Space(\text{ llog }) . \end{array}$$

The proof of these properties is straightforward using the fact that $L_\mathcal{F} \in \Pi_2 Space(\text{ llog })$ and $\overline{L}_\mathcal{F} \in \Sigma_2 Space(\text{ llog })$. The separation

Theorem 8 For any $k \geq 2$ holds

$$\begin{array}{rcl} L_{\Sigma k} & \notin & \Pi_k Space(o(\log)) , \\ L_{\Pi k} & \notin & \Sigma_k Space(o(\log)) . \end{array}$$

We will define specific inputs that belong to $L_{\Sigma k}$ and $L_{\Pi k}$ and show that any sublogarithmic space-bounded machine cannot work correctly on both inputs.

Let $L = L_\mathcal{F}$ be fixed. Recall that infinitely many $n \in \mathbb{N}$ exist with $n \in \mathcal{F}$, $1^n \in L$ and $1^{n+n!} \notin L$.

Definition 7 For $n \in \mathcal{F}$ define words

$$W^n_{\Sigma 2} \;:=\; 1^{n+n!} \quad \text{and} \quad W^n_{\Pi 2} \;:=\; 1^n ,$$

and for $k \geq 3$

$$\begin{array}{rcl} W^n_{\Sigma k} & := & \left[W^n_{\Sigma k-1}\; 0\right]^{m_{k,n}} \; W^n_{\Pi k-1}\; 0 \; \left[W^n_{\Sigma k-1}\; 0\right]^{m_{k,n}} , \\[2ex] W^n_{\Pi k} & := & \left[W^n_{\Sigma k-1}\; 0\right]^{m_{k,n}} \; W^n_{\Sigma k-1}\; 0 \; \left[W^n_{\Sigma k-1}\; 0\right]^{m_{k,n}} , \end{array}$$

where the $m_{k,n}$ are the parameters already used in the Position-Shift-Lemma.

From the definition follows easily

Lemma 13 For $k \geq 2$ and every $n \in \mathcal{F}$

$$\begin{array}{lcl} W^n_{\Sigma k} \in L_{\Sigma k} & \text{and} & W^n_{\Sigma k} \notin L_{\Pi k} \ , \\ W^n_{\Pi k} \in L_{\Pi k} & \text{and} & W^n_{\Pi k} \notin L_{\Sigma k} \ . \end{array}$$

Let $k \geq 2$ and $S \in$ SUBLOG be a space bound. We will prove Theorem 8 by showing that if a Σ_k TM M accepts $L_{\Pi k}$ in space S then for sufficiently large $n \in \mathcal{F}$ M accepts $W^n_{\Sigma k}$, too. Similarly, if a Π_k TM M accepts $L_{\Sigma k}$ in space S then for large $n \in \mathcal{F}$ it accepts $W^n_{\Pi k}$ and hence makes a mistake. Recall that $\mathcal{N}_{M,S}$ denotes the constant defined for M and S in Section 2.

Proposition 1 Let $S \in o(\log)$ and M be an ATM. Then for any $k \geq 2$, for all $n \geq \mathcal{N}_{M,S}$, for all strings $U, V \in \{0,1\}^*$, and for any configuration (α, i) with

1. $i \leq |U|$ or $i > |U\ W^n_{\Pi k}|$ and
2. $Space_M(\alpha, i,\ U\ W^n_{\Pi k}\ V) \leq S(n)$ and $Space_M(\alpha, \hat{\imath},\ U\ W^n_{\Sigma k}\ V) \leq S(n)$

holds:

$$\begin{array}{lcll} \mathtt{acc}^k_M(\alpha, i, U\ W^n_{\Pi k}\ V) & \Longrightarrow & \mathtt{acc}^k_M(\alpha, \hat{\imath}, U\ W^n_{\Sigma k}\ V) & \text{if } (\alpha,i) \text{ is existential,} \\ \mathtt{acc}^k_M(\alpha, \hat{\imath}, U\ W^n_{\Sigma k}\ V) & \Longrightarrow & \mathtt{acc}^k_M(\alpha, i, U\ W^n_{\Pi k}\ V) & \text{if } (\alpha,i) \text{ is universal.} \end{array}$$

Proof. Remember that $\hat{\imath}$ was defined as

$$\hat{\imath} \ = \ \begin{cases} i & \text{if } i \leq |U|, \\ i + (|W^n_{\Sigma k}| - |W^n_{\Pi k}|) & \text{if } i > |U\ W^n_{\Pi k}|. \end{cases}$$

For $k = 2$ the implications above follow from the 1-Alternation Lemma.

To establish the proposition for $k > 2$ we consider the first time when the machine M makes an alternation and inductively use the corresponding properties for the strings $W^n_{\Sigma k-1}$ and $W^n_{\Pi k-1}$. The argument concentrates only on the block in the middle of a $W^n_{\Sigma k}$ string, which is a $W^n_{\Pi k-1}$ word, and analogously for $W^n_{\Pi k}$ strings with a $W^n_{\Sigma k-1}$ word in the middle. The main technical difficulty for the following argument is the possibility that in an accepting computation the machine may just make its first alternation in the middle block, and therefore may notice the difference between the $W^n_{\Sigma k}$ and $W^n_{\Pi k}$ strings. But the Configuration- and Position-Shift-Lemmata imply that there also exist accepting computations with the first alternation outside this critical region.

The details are as follows. Assume that the configuration (α, i) fulfills properties 1. and 2. Let $n \geq \mathcal{N}_{M,S}$, and define

$$\begin{array}{rcll} X & := & U\ W^n_{\Pi k}\ V \ = \ U'\ W^n_{\Sigma k-1}\ V' \ , & \\ Y & := & U\ W^n_{\Sigma k}\ V \ = \ U'\ W^n_{\Pi k-1}\ V', & \text{where} \\ U' & := & U\ \left[W^n_{\Sigma k-1}\ 0\right]^{m_{k,n}} & \text{and} \\ V' & := & 0\ \left[W^n_{\Sigma k-1}\ 0\right]^{m_{k,n}}\ V \ , & \\ \Delta & := & |W^n_{\Sigma k-1}\ 0| \cdot n! \ , & \\ \tilde{\jmath} & := & \begin{cases} j & \text{if } j \leq |U'| \ , \\ j + (|W^n_{\Pi k-1}| - |W^n_{\Sigma k-1}|) & \text{if } j > |X| - |V'|. \end{cases} & \end{array}$$

Note that $\hat{\imath}$ is defined with respect to the partition of the inputs X, Y with the prefix U and the suffix V, where $\tilde{\jmath}$ is taken with respect to the prefix U' and suffix V'. Since

$$|W^n_{\Pi k-1}| - |W^n_{\Sigma k-1}| \ = \ |W^n_{\Sigma k}| - |W^n_{\Pi k}|$$

$$\hat{\imath} \;=\; \tilde{\imath} \qquad \text{whenever both values are defined.} \tag{i}$$

First we prove the following

Claim 1 For any memory state $|\alpha_1| \leq |\alpha_2| \leq S(n)$ and all j_1, j_2 with

$$j_1, j_2 \in \; [0 \;\ldots\; |U'|] \;\cup\; [|U' \, W^n_{\Sigma k-1}| + 1 \;\ldots\; |X| + 1]$$

holds:

$$(\alpha_1, j_1) \models_{M,X} (\alpha_2, j_2) \qquad \Longleftrightarrow \qquad (\alpha_1, \tilde{j}_1) \models_{M,Y} (\alpha_2, \tilde{j}_2) \; .$$

Proof. For suitable $Z_1, Z_2 \in \{0,1\}^*$ the words considered can be written as

$$W^n_{\Sigma k} \;=\; Z_1 \, 1^n \, Z_2 \quad \text{and} \quad W^n_{\Pi k} \;=\; Z_1 \, 1^{n+n!} \, Z_2 \qquad \text{if } k \text{ is odd, and}$$
$$W^n_{\Sigma k} \;=\; Z_1 \, 1^{n+n!} \, Z_2 \quad \text{and} \quad W^n_{\Pi k} \;=\; Z_1 \, 1^n \, Z_2 \qquad \text{for even } k.$$

The claim then follows from the Pumping Lemma (Lemma 3). ∎

A.) First we consider existential configurations (α, i). Assume that

$$\mathtt{acc}^k_M(\alpha, i, X)$$

is true. Hence there exists an existential computation path from (α, i) to a final or universal configuration (β, j):

$$(\alpha, i) \models_{M,X} (\beta, j) \tag{ii}$$

with the property

$$\mathtt{acc}^{k-1}_M(\beta, j, X) \; . \tag{iii}$$

We may assume that

$$j \leq |U'| \quad \text{or} \quad j > |U' \, W^n_{\Sigma k-1}| \; , \tag{iv}$$

because if $|U'| < j \leq |U' \; W^n_{\Sigma k-1}|$ then for $Z_1 := U$, $Z_2 := V$, $W := W^n_{\Sigma k-1} \; 0$, and $s := 2m_{k,n} + 1 - n - (n + n!)$ the Configuration-Shift-Lemma implies

$$(\alpha, i) \models_{M,X} (\beta, j - \Delta) \; .$$

Moreover, for $r := t := m_{k,n}$ and for $s := 1$, from the Position-Shift-Lemma we can deduce

$$\mathtt{acc}^{k-1}_M(\beta, j - \Delta, X) \; .$$

Therefore, if $|U'| < j \leq |U' \; W^n_{\Sigma k-1}|$ the configuration (β, j') with $j' := j - \Delta$ instead of (β, j) satisfies properties (ii)-(iv).

Since $\hat{\imath} = \tilde{\imath}$ according to (i), Claim 1 applied to (ii) yields

$$(\alpha, \hat{\imath}) \;=\; (\alpha, \tilde{\imath}) \models_{M,Y} (\beta, \tilde{j}) \; .$$

A terminating configuration (β, j) must be accepting because of (ii) and (iii), hence $(\beta, \tilde{j})$ is accepting and $\mathtt{acc}^k_M(\alpha, \hat{\imath}, Y)$ is true.

For a universal (β, j) we apply the induction hypothesis. Because of (iv) the requirements 1. and 2. of the proposition are fulfilled for $k - 1$ and $i := \tilde{j}$. Property (i) implies for this choice of i that $\hat{\imath} = j$. Therefore, in (iii) replacing j by $\hat{\imath}$ one can conclude

$$\mathtt{acc}^{k-1}_M(\beta, \hat{\imath}, X) \qquad \Longrightarrow \qquad \mathtt{acc}^{k-1}_M(\beta, i, Y) \;=\; \mathtt{acc}^{k-1}_M(\beta, \tilde{j}, Y) \; .$$

Hence, we can conclude that $\mathtt{acc}_M^k(\alpha,\hat{\imath},Y)$ holds. This proves the proposition for existential configurations.

B.) Now let us consider universal configurations (α,i), for which $\mathtt{acc}_M^k(\alpha,\hat{\imath},Y)$ holds. We have to show that $\mathtt{acc}_M^k(\alpha,i,X)$ is true.

Claim 2 For input X any universal computation path starting in (α,i) is finite.

Proof. Assume, to the contrary, that there exists an infinite computation path which is universal and starts in (α,i). This means that there exists a universal configuration (β,j) such that

$$(\alpha,i) \models_{M,X} (\beta,j) \models_{M,X} (\beta,j)\ . \tag{v}$$

We can assume that

$$j \le |U'| \quad \text{or} \quad j > |U'\ W^n_{\Sigma k-1}| \tag{vi}$$

because if $|U'| < j \le |U'\ W^n_{\Sigma k-1}|$ the Configuration-Shift-Lemma implies

$$(\alpha,i) \models_{M,X} (\beta,j-\Delta) \models_{M,X} (\beta,j-\Delta)\ .$$

Hence, (v) and (vi) are fulfilled for $j' := j-\Delta$. Form (i) and Claim 1 follows

$$(\alpha,\hat{\imath}) \ = \ (\alpha,\tilde{\imath}) \models_{M,Y} (\beta,\tilde{\jmath}) \models_{M,Y} (\beta,\tilde{\jmath})\ .$$

This means that for input Y there exists an infinite computation path, which is universal and starts in $(\alpha,\hat{\imath})$. We get a contradiction to $\mathtt{acc}_M^k(\alpha,\hat{\imath},Y)$. ∎

Now we want to show that for any final or existential configuration (β,j) that can be reached from (α,i) on a universal computation path holds

$$\mathtt{acc}_M^{k-1}(\beta,j,X)\ .$$

According to Claim 2 this proves $\mathtt{acc}_M^k(\alpha,i,X)$. Let $(\alpha,i) \models_{M,X} (\beta,j)$. Two cases will be distinguished.

Case 1. $j \le |U'|$ or $j > |U'\ W^n_{\Sigma k-1}|$.

From Claim 1 it follows that

$$(\alpha,\hat{\imath}) \ = \ (\alpha,\tilde{\imath}) \models_{M,Y} (\beta,\tilde{\jmath})\ .$$

The assumption $\mathtt{acc}_M^k(\alpha,\hat{\imath},Y)$ implies

$$\mathtt{acc}_M^{k-1}(\beta,\tilde{\jmath},Y) \tag{vii}$$

For a final configuration (β,j) one can conclude from property (vii) that β must be accepting, hence $\mathtt{acc}_M^{k-1}(\beta,j,X)$ holds.

For an existential (β,j) the same implication holds using the induction hypothesis.

Case 2. $|U'| < j \le |U'\ W^n_{\Sigma k-1}|$.

The Configuration-Shift-Lemma implies

$$(\alpha,i) \models_{M,X} (\beta,j-\Delta)\ .$$

In the proof of Case 1 it was shown for the configuration $(\beta,j-\Delta)$ that

$$\mathtt{acc}_M^{k-1}(\beta,j-\Delta,X)$$

holds. Using the Position-Shift-Lemma we obtain $\mathtt{acc}_M^{k-1}(\beta,j,X)$. This completes the proof of Proposition 1. ∎

Next, we will show that the second requirement of the proposition above is always fulfilled.

Proposition 2 Let $k \geq 2$ and M be an ATM of space complexity S with $S \in o(\log)$. Then there exists a bound $S' \in o(\log)$ such that for all $n \geq \mathcal{N}_{M,S'}$

$$Space_M(W^n_{\Pi k}) \leq S'(n) \quad \text{and} \quad Space_M(W^n_{\Sigma k}) \leq S'(n) \ .$$

Proof. The idea of the proof is as follows. If in $W^n_{\Pi k}$ and $W^n_{\Sigma k}$ all substrings generated in the recursive construction which are multiplies of $n!$, are cancelled, then the remaining word has a length $p_k(n)$, which is polynomial in n. Using the Small-Space-Bound-Lemma, which shows that a sublogarithmic space-bounded machine M does not notice a difference when an arbitrary block of the input is added $n!$ times, it follows that M must obey a space bound $S(p_k(n))$ on $W^n_{\Pi k}$ and $W^n_{\Sigma k}$. If S grows sublogarithmically in n so does $S(p_k(n))$.

Below the technical details of this proof are outlined. Let

$$V^1_2(n) := 1^n \ .$$

For $d \geq 3$ define

$$V^1_d(n) \ := \ \left[V^1_{d-1}(n)\ 0\right]^{2dn+1} ,$$

and for $i = 2, \ldots, d-1$

$$V^i_d(n) \ := \ \left[V^{i-1}_{d-1}(n)\ 0\right]^{2m_{d,n}+1} .$$

Define also a sequence of polynomials $p_d(n)$ as follows:

$$p_2(n) := n \qquad \text{and for } d \geq 3 \qquad p_d(n) \ := \ (2dn+1) \cdot (p_{d-1}(n) + 1) \ .$$

Obviously, for any $d \geq 2$ and for all n

$$p_d(n) = |V^1_d(n)| \ .$$

Let M be an ATM of space complexity S with $S \in o(\log)$. Define $S'(n) := S(p_k(n))$. Obviously, $S' \in o(\log)$. Let n be an integer with $n \geq \mathcal{N}_{M,S'}$.

Since M is S space-bounded

$$Space_M(V^1_k(n)) \ \leq \ S(p_k(n)) \ = \ S'(n) \ . \tag{i}$$

It is easy to check that for any n and for any $i \in [1...k-2]$ there are words $Z_1, Z_2, \ldots, Z_r$ over the alphabet $\{0\}$, where

$$r \ := \prod_{t=k-i+2}^{k} 2m_{t,n} + 1 \ ,$$

(for $i = 1$ take $r := 1$), such that for $W := V^1_{k-i}(n)\ 0$, $a := 2n(k-i)+n+1$ and $b := 2(k-i+1)$:

$$\begin{aligned} V^i_k(n) \ &= \ W^{a+n}\ Z_1\ W^{a+n}\ Z_2 \ldots Z_{r-1}\ W^{a+n}\ Z_r \ , \\ V^{i+1}_k(n) \ &= \ W^{a+n+bn!}\ Z_1\ W^{a+n+bn!}\ Z_2 \ldots Z_{r-1}\ W^{a+n+bn!}\ Z_r \ . \end{aligned}$$

By the Small-Space-Bound-Lemma the following implications hold for $i = 1, 2, \ldots, k-2$

$$Space_M(V^i_k(n)) \ \leq \ S'(n) \qquad \Longrightarrow \qquad Space_M(V^{i+1}_k(n)) \ \leq \ S'(n).$$

Therefore, by (i), we obtain that

$$Space_M(V^{k-1}_k(n)) \ \leq \ S'(n) \ . \tag{ii}$$

Now let $\hat{W}^n_{\Sigma k}$ denote a word $W^n_{\Sigma k}$ where all substrings $1^{n+n!}$ are reduced to 1^n. Similarly, $\hat{W}^n_{\Pi k}$ is obtained from $W^n_{\Pi k}$. Obviously, by the Small-Space-Bound-Lemma, $Space_M(\hat{W}^n_{\Sigma k}) \leq S'(n)$ implies

$Space_M(W^n_{\Sigma k}) \le S'(n)$ and $Space_M(\hat{W}^n_{\Pi k}) \le S'(n)$ implies $Space_M(W^n_{\Pi k}) \le S'(n)$. The proposition holds since

$$\hat{W}^n_{\Sigma k} \;=\; \hat{W}^n_{\Pi k} \;=\; V_k^{k-1}(n)$$

and by (ii) the space used by M on input $V_k^{k-1}(n)$ is bounded by $S'(n)$. ∎

Now we are ready to prove Theorem 8. Let us assume that M is a Σ_k TM accepting $L_{\Pi k}$ in sublogarithmic space S . By Proposition 2 there exists a function $S' \in o(\log)$ such that for any $n \ge \mathcal{N}_{M,S'}$

$$Space_M(W^n_{\Pi k}) \le S'(n) \quad \text{and} \quad Space_M(W^n_{\Sigma k}) \le S'(n) \ .$$

Let n with $n \in \mathcal{F}$ be an integer larger than $\mathcal{N}_{M,S'}$ (such an n exists since $\mathcal{F}$ is infinite). By Lemma 13 $W^n_{\Pi k} \in L_{\Pi k}$, hence M has to accept $W^n_{\Pi k}$, which means that $\mathtt{acc}^k_M(\alpha_0, 0, W^n_{\Pi k})$ is true, where $(\alpha_0, 0)$ is the initial configuration of M . From Proposition 1 we conclude that $\mathtt{acc}^k_M(\alpha_0, 0, W^n_{\Sigma k})$ holds, too, and hence M accepts $W^n_{\Sigma k}$, which by Lemma 13 does not belong to $L_{\Pi k}$ – a contradiction.

In the same way one shows that if M is a Π_k TM that accepts $L_{\Sigma k}$ in space S then M accepts $W^n_{\Pi k}$. ∎

3.3 Unbounded Number of Alternations

Let us now consider ATMs with a nonconstant bounding function A for the number of alternations. The separating results for A-alternation-bounded space classes (Theorem 2) follow from the propositions below.

Definition 8 Let $A : \mathbb{N} \to \mathbb{N}$ be a function with $A(n) \ge 2$ for all n and define

$$\begin{aligned} L_\Sigma(A) &:= \{X \mid X = W0^r \text{ for some } r \in \mathbb{N} \text{ and } W \in L_{\Sigma k} \text{ for some } k \le A(|X|)\} \ , \\ L_\Pi(A) &:= \{X \mid X = W0^r \text{ for some } r \in \mathbb{N} \text{ and } W \in L_{\Pi k} \text{ for some } k \le A(|X|)\} \ . \end{aligned}$$

Lemma 14 For any $S \in \mathtt{SUBLOG}$ and all functions $A \ge 2$ computable in space S holds:

$$\begin{aligned} L_\Sigma(A) &\in \Sigma_A Space(S) \ , \\ L_\Pi(A) &\in \Pi_A Space(S) \ . \end{aligned}$$

Proof. On input $X = W0^r$ the machine first computes $a := A(|X|)$ and initializes a counter with that value. It remains to check whether $W \in L_{\Sigma k}$ for some $k \le a$. This can be done similarly as in the case for fixed k , decrementing the counter each time an alternation has been performed. ∎

For functions $A, B : \mathbb{N} \to \mathbb{N}$ let $A \le_* B$ denote that $A(m) \le B(m)$ for all $m \in \mathbb{N}$ with equality for infinitely many m .

Proposition 3 For any $S \in \mathtt{SUBLOG}$ and for all functions A and B with $1 < A \le_* B$ and $B \cdot S \in o(\log)$ holds:

$$\begin{aligned} L_\Sigma(A) &\notin \Pi_B Space(S) \ , \\ L_\Pi(A) &\notin \Sigma_B Space(S) \ . \end{aligned}$$

Proof. Let $S \in \mathtt{SUBLOG}$ and let A, B be functions with $1 < A \le_* B$ and $B \cdot S \in o(\log)$. These assumptions imply that there exists a constant $m_0 \ge \exp\exp 9$ such that $A(m) < \frac{\log m}{\operatorname{llog} m}$ for all $m \ge m_0$. Define functions h and f as follows

$$\begin{aligned} h(m) &:= \frac{\exp\left(\frac{\log m}{A(m)}\right)}{3\ A(m)} \ , \\ f(m) &:= \max\{\ell \mid \ell \in \mathcal{F} \cup \{0\},\ \ell \le h(m)\} \ . \end{aligned}$$

For $m \geq m_0$ we can bound h by

$$h(m) \;\leq\; \exp\left(\frac{\log m}{2}\right) \;=\; m^{1/2} \;,$$
$$h(m) \;\geq\; \frac{\exp\ \text{llog}\ m}{3 \log m \;/\; \text{llog}\ m} \;=\; \frac{\text{llog}\ m}{3} \;\geq\; 3 \;.$$

and hence $f(m) \in \mathcal{F}$. Moreover, from lemma 10 follows

$$f(m) \;\geq\; h(m)^{1/3} \;\geq\; \left(\frac{1}{3}\ \text{llog}\ m\right)^{1/3} \;. \tag{i}$$

Define the function $S' : \mathbb{N} \to \mathbb{N}$ as follows

$$S'(n) \;:=\; \max\Big(\{0\} \;\cup\; \{S(m) \mid f(m) = n\}\Big) \;.$$

Because f grows unboundedly $S'(n)$ will always be a finite number.

Lemma 15 $S' \in o(\log)$.

Proof. First we show that $S \in o(\log \circ f)$. By assumption,

$$S \in o\left(\frac{\log}{A}\right) \qquad \text{and} \qquad \log A \;\leq\; \text{llog}\ m \;\leq\; S \;.$$

This implies

$$S \in o\left(\frac{\log}{A} - S\right) \;=\; o\left(\frac{\log}{A} - \log A\right) \;=\; o(\log h) \;=\; o(\log f) \;.$$

Thus, if n goes to ∞

$$\frac{S(n)}{\log f(n)} \;\to\; 0$$

and

$$\frac{S'(n)}{\log n} \;=\; \max_{\{m \mid f(m)=n\}} \frac{S(m)}{\log n} \;=\; \max_{\{m \mid f(m)=n\}} \frac{S(m)}{\log f(m)} \;.$$

If n goes to ∞ also m has to do this, and hence all quotients converge to 0. But this means that $S' \in o(\log)$. ∎

Consider the function t defined by

$$t(m) \;:=\; m - p_{A(m)}(f(m))$$

where $p_d(n)$ has already been defined in the proof of Proposition 2, and note that

$$p_{A(m)}(f(m)) \;\leq\; (3\ A(m)\ f(m))^{A(m)} \;\leq\; m \;.$$

Thus, $t(m) \geq 0$.

Now let M be an ATM that works in space $S(|X|)$ and makes at most $B(|X|) - 1$ alternations. Let m be an integer with

$$m \;\geq\; \max\{m_0,\ \exp\exp 3(\mathcal{N}_{M,S'})^4\} \qquad \text{and} \qquad A(m) = B(m) \tag{ii}$$

Such an m exists since $A \leq_* B$. Then define

$$k := A(m) \quad \text{and} \quad n := f(m) \;.$$

By (i) and (ii) $n \geq \mathcal{N}_{M,S'}$. Moreover, $n \in \mathcal{F}$ and M makes no more then $k-1$ alternations on any input of length m. Let

$$X \;:=\; V_k^1(n)\ 0^{t(m)}$$

with the word $V_k^1(n)$ defined as in the proof of Proposition 2. Since the length of $V_k^1(n)$ is $p(k,n)$ the string X is of length m. From the definition of S' follows that

$$\begin{array}{rcl} Space_M(X) & \leq & S(m) \leq \max\{S(m') \mid f(m') = n\} = S'(n) \qquad \text{and} \\ Alter_M(X) & \leq & B(m) - 1 \leq \exp S(m) \leq \exp S'(n) . \end{array}$$

Hence, for the machine M and the function S' the assumptions of the Small-Space-Bound-Lemma and the Small-Alternation-Bound-Lemma are fulfilled. Using the Small-Space-Bound-Lemma for the input X in the similar way as in the proof of Proposition 2 one can show that

$$Space_M(W^n_{\Sigma k}\, 0^{t(m)}) \ , \ Space_M(W^n_{\Pi k}\, 0^{t(m)}) \ = \ Space_M(X) \leq S'(n) \ .$$

Similarly, by the Small-Alternation-Bound-Lemma one obtains that

$$Alter_M(W^n_{\Sigma k}\, 0^{t(m)}) \ , \ Alter_M(W^n_{\Pi k}\, 0^{t(m)}) \ = \ Alter_M(X) \leq B(m) - 1 = k - 1 \ .$$

Now we can finish the proof. Let us assume that M is a Σ_B TM accepting $L_\Pi(A)$ in space S. By Lemma 13 holds $W^n_{\Pi k} \in L_{\Pi k}$, hence M has to accept $W^n_{\Pi k}\, 0^{t(m)}$. But this means that $\mathtt{acc}^k_M(\alpha_0, 0, W^n_{\Pi k}\, 0^{t(m)})$ is true, where $(\alpha_0, 0)$ is the initial configuration of M. From Proposition 1 we conclude that $\mathtt{acc}^k_M(\alpha_0, 0, W^n_{\Sigma k}\, 0^{t(m)})$ holds, too. Therefore M accepts $W^n_{\Sigma k}\, 0^{t(m)}$, which by Lemma 13 does not belong to $L_\Pi(A)$ – a contradiction.

In the same way, one can show that if M is a Π_B TM that accepts $L_\Sigma(A)$ in space S then M accepts $W^n_{\Pi k}\, 0^{t(m)}$. ∎

4 Closure Properties

In this section we discuss closure properties of $\Sigma_k Space(S)$ and $\Pi_k Space(S)$ classes for sublogarithmic bounds S. First for any integer $k \geq 2$ we define the languages

$$A_{\Sigma k} := L_k\,\{0\}\,L_{\Sigma k} \ , \quad B_{\Sigma k} := L_{\Sigma k}\,\{0\}\,L_k \ ,$$

and symmetrically

$$A_{\Pi k} := L_k\,\{0\}\,L_{\Pi k} \ , \quad B_{\Pi k} := L_{\Pi k}\,\{0\}\,L_k \ .$$

It is easy to see that

$$A_{\Sigma k}, B_{\Sigma k} \in \Sigma_k Space(\text{ llog }) \quad \text{and} \quad A_{\Pi k}, B_{\Pi k} \in \Pi_k Space(\text{ llog }) \ . \qquad \text{(i)}$$

Proposition 4 For all $k \geq 2$ holds:

$$\begin{array}{rcl} A_{\Sigma k} \cap B_{\Sigma k} & \in & \Pi_{k+1} Space(\text{ llog }) \setminus \Sigma_{k+1} Space(o(\log)) \ , \\ A_{\Pi k} \cup B_{\Pi k} & \in & \Sigma_{k+1} Space(\text{ llog }) \setminus \Pi_{k+1} Space(o(\log)) \ . \end{array}$$

Proof. It is well known that for any function S the classes $\Sigma_k Space(S)$ are closed under union, and symmetrically the $\Pi_k Space(S)$ are closed under intersection (see e.g. [25]). Hence by (i), $A_{\Sigma k} \cap B_{\Sigma k} \in \Pi_{k+1} Space(\text{ llog })$ and $A_{\Pi k} \cup B_{\Pi k} \in \Sigma_{k+1} Space(\text{ llog })$. To prove that $A_{\Sigma k} \cap B_{\Sigma k} \notin \Sigma_{k+1} Space(o(\log))$ and $A_{\Pi k} \cup B_{\Pi k} \notin \Pi_{k+1} Space(o(\log))$ first we modify Proposition 2 in the following way:

Proposition 2′ Let $k \geq 2$ and M be an ATM of space complexity S with $S \in o(\log)$. Then there exists a bound $S'' \in o(\log)$ such that for all $n \geq \mathcal{N}_{M,S''}$ and words $W_1, W_2 \in \{W^n_{\Sigma k}\ ,\ W^n_{\Pi k}\}$

$$Space_M(W_1\, 0\, W_2) \leq S''(n) \ .$$

Proof. Let $S''(n) := S(2p_k(n)+1)$, where p_k is the polynomial specified in the proof of Proposition 2. It is easy to check that the proof of Proposition 2 generalizes to this situation. ∎

Let us assume, to the contrary, that $A_{\Sigma k} \cap B_{\Sigma k} \in \Sigma_{k+1}Space(S)$, for some $S \in o(\log n)$. Let M be an S space-bounded Σ_{k+1} TM for $A_{\Sigma k} \cap B_{\Sigma k}$. Choose $n \in \mathcal{F}$ sufficiently large. By Lemma 13 $W^n_{\Sigma k} \in L_{\Sigma k}$ hence M has to accept

$$X = W^n_{\Sigma k}\, 0\, W^n_{\Sigma k}$$

which means that there exists an existential computation path starting in initial configuration $(\alpha_0, 0)$ and ending in a universal configuration (β, j), with

$$(\alpha_0, 0) \models_{M,X} (\beta, j) , \tag{ii}$$

and

$$\mathtt{acc}^k_M(\beta, j, X) . \tag{iii}$$

(The trivial case that M accepts X without alternation could be handled similarly.) Now let $Y_1 := W^n_{\Sigma k}\, 0\, W^n_{\Pi k}$ and $Y_2 := W^n_{\Pi k}\, 0\, W^n_{\Sigma k}$. By Proposition $2'$ there exists $S'' \in o(\log n)$ such that

$$Space_M(X),\ Space_M(Y_1),\ Space_M(Y_2) \quad \leq \quad S''(n) .$$

Therefore, applying Claim 1 (from the Proof of Proposition 1) and Proposition 1 to (ii) and (iii), resp., we obtain

$$(\alpha_0, 0) \models_{M,Y_1} (\beta, j) \quad \text{and} \quad \mathtt{acc}^k_M(\beta, j, Y_1)$$

if $j \leq |W^n_{\Sigma k}\, 0|$ and otherwise

$$(\alpha_0, 0) \models_{M,Y_2} (\beta, \hat{j}) \quad \text{and} \quad \mathtt{acc}^k_M(\beta, \hat{j}, Y_2) ,$$

where $\hat{j} = j + |Y_2| - |X|$. Hence M also accepts input Y_1 or Y_2. This yields a contradiction since, by Lemma 13, $Y_1, Y_2 \notin A_{\Sigma k} \cap B_{\Sigma k}$.

Similarly, one can show that if a Π_{k+1} TM accepts $A_{\Pi k} \cup B_{\Pi k}$ within space $S \in o(\log n)$, then it has to reject X, but it also rejects input Y_1 or Y_2, which both belong to $A_{\Pi k} \cup B_{\Pi k}$ – a contradiction!

∎

This result can be applied to **prove Theorem 3:**

For all $k \geq 2$ and any $S \in \mathtt{SUBLOG}$ holds:

1. $\Sigma_k Space(S)$ and $\Pi_k Space(S)$ are not closed under complementation.
2. $\Sigma_k Space(S)$ is not closed under intersection,
3. $\Pi_k Space(S)$ is not closed under union.
4. $\Sigma_k Space(S)$ and $\Pi_k Space(S)$ are not closed under concatenation.

(1) follows immediately from Lemma 12, Theorem 8 and the following equations: $L_{\Sigma k} = L_k \cap \overline{L}_{\Pi k}$, and $L_{\Pi k} = L_k \cap \overline{L}_{\Sigma k}$, where L_k is the regular language introduced in Definition 6.

By (i) $A_{\Sigma k}, B_{\Sigma k} \in \Sigma_k Space(\text{ llog})$ and $A_{\Pi k}, B_{\Pi k} \in \Pi_k Space(\text{ llog})$. On the other hand, from Proposition 4 $A_{\Sigma k} \cap B_{\Sigma k} \notin \Sigma_{k+1} Space(o(\log))$ and $A_{\Pi k} \cup B_{\Pi k} \notin \Pi_{k+1} Space(o(\log))$. This proves (2) and (3).

Property (4) for Σ_k classes follows from the fact that for any $k \geq 2$ $L_{\Sigma k}\, \{0\}\, L_{\Sigma k} = A_{\Sigma k} \cap B_{\Sigma k}$ does not belong to $\Sigma_k Space(o(\log))$, but $L_{\Sigma k} \in \Sigma_k Space(\text{ llog})$. To see that $\Pi_k Space(S)$ is not closed under concatenation define the languages

$$L^1_k := L_k \cup \{\varepsilon\}$$

where ε denotes the empty string and

$$L_k^2 := \{w_1 0 w_2 0 \dots 0 w_p 0 \mid p \in \mathbb{N},\ w_i \in L_{k-1} \text{ and } w_1 \in L_{\Pi k-1}\} .$$

Obviously, both languages belong to $\Pi_k Space(\text{ llog })$, but from Theorem 8 follows

$$L_k^1 L_k^2 = L_{\Sigma k} \notin \Pi_k Space(o(\log)) .$$

∎

5 Lower Space Bounds for Context-Free Languages

Proposition 5 $L_{\neq} = \{1^n 0 1^m : n \neq m\} \notin ASpace(o(\log))$.

Proof. Let us assume, to the contrary, that $L_{\neq}$ is recognized by an S space-bounded ATM A for some $S \in o(\log)$. Let $S'(n) := S(2n+1)$. Obviously, $S' \in o(\log)$. Let $\hat{n} := \mathcal{N}_{M,S'}$. Then by the Small-Space-Bound-Lemma for all $k, \ell \geq 0$

$$Space_A(1^{\hat{n}} 0 1^{\hat{n}}) = Space_A(1^{\hat{n}+k\hat{n}!} 0 1^{\hat{n}+\ell\hat{n}!}). \qquad \text{(i)}$$

Let

$$\hat{s} = Space_A(1^{\hat{n}} 0 1^{\hat{n}}). \qquad \text{(ii)}$$

For this fixed $\hat{n}$ we define the following language $\hat{L} = \{1^{\hat{n}+k\hat{n}!} 0 1^{\hat{n}+\ell\hat{n}!} : k, \ell \in \mathbb{N} \text{ and } k \neq \ell\}$, and construct an automaton $\hat{A}$ that recognizes $\hat{L}$. $\hat{A}$ performs the following algorithm:

Step 1. Check deterministically if the input X has the form $1^{\hat{n}+k\hat{n}!} 0 1^{\hat{n}+\ell\hat{n}!}$ for some integers k and ℓ; reject and stop if this condition does not hold;

Step 2. Move the head to the first symbol of the input and start to simulate the machine A.

It is obvious that $\hat{A}$ accepts an input $X = 1^{\hat{n}+k\hat{n}!} 0 1^{\hat{n}+\ell\hat{n}!}$ if and only if A accepts X. Hence we have $L(\hat{A}) = \hat{L}$. It is easy to see that step 1 can be performed within space $O(\log \hat{n}!)$, which is a constant. Moreover from (i) and (ii) it follows that step 2 also requires only constant space $\hat{s}$. Hence $\hat{A}$ recognizes $\hat{L}$ within constant space. We get a contradiction, since $\hat{L}$ is non-regular. ∎

Using a similar proof one can show that the language

$$L_{=} := \{1^n 0 1^n : n \in \mathbb{N}\}$$

is not in $ASpace(o(\log))$, too.

The rest of this section is devoted to the lower space bounds for a large subset of nonregular context-free languages.

The block structure of a bounded language L can equivalently be represented using a finite alphabet $\{a_1, \dots, a_r\}$. Then L is a subset of $\{a_1\}^* \dots \{a_r\}^*$.

Definition 9 Let $V(L)$ denote the set $\{(v_1, \dots, v_r) \in \mathbb{N}^r \mid a_1^{v_1} \dots a_r^{v_r} \in L\}$. Sets of the form $\{\alpha + n_1\beta_1 + \dots + n_k\beta_k \mid n_1, \dots, n_k \in \mathbb{N}\}$ with $\alpha, \beta_1, \dots, \beta_k \in \mathbb{N}^r$, are called *linear sets.* A finite union of linear sets is a *semilinear set.* A language L is *semilinear* if $L \subseteq \{a_1\}^* \dots \{a_r\}^*$ and $V(L)$ is a semilinear set.

Proposition 6 Let $L \subseteq \{a_1\}^* \dots \{a_r\}^*$ be semilinear and let $L, \overline{L} \in ASpace(S)$ for some $S \in o(\log)$. Then L is regular.

Proof. For $r = 1$ the proposition is true because every semilinear tally language is regular. Let us assume that $r > 1$ and that the proposition holds for $r-1$. Sets of the form

$$\{\alpha + q_1\gamma_1 + \ldots + q_k\gamma_k \mid q_1, \ldots, q_k \in \mathbb{R}_+\}$$

with $\gamma_1, \ldots, \gamma_k \in \mathbb{N}^r$ are called *cones* (see [1]). Assume now, to the contrary, that L is nonregular. To show that this cannot occur we first construct a semilinear language $\tilde{L} \in ASpace(S)$ that is also nonregular and for which there exists an r-dimensional cone C such that $V(\tilde{L}) \cap C = \emptyset$. To this end, methods developed by Alt and Mehlhorn in [1], [4] will be used.

Lemma ([1]) There exists an r-dimensional cone C and a regular language $R \subseteq \{a_1\}^* \ldots \{a_r\}^*$ with

$$V(L) \cap C \;=\; V(R) \cap C\ .$$

Let R and C be as in the lemma. Define $L_1 := L \setminus R$ and $L_2 := R \setminus L$. Obviously L_1 or L_2 is nonregular since L is nonregular. We set $\tilde{L} := L_1$ if L_1 is nonregular and $\tilde{L} := L_2$ otherwise. The language $\tilde{L}$ is semilinear since the class of semilinear sets is closed under Boolean operations ([12]). Moreover, $\tilde{L} \in ASpace(S)$, because $L, \overline{L} \in ASpace(S)$ and $V(\tilde{L}) \cap C = \emptyset$ for the r-dimensional cone C.

Definition 10 Let us call a set $K \subseteq \mathbb{N}^r$ extended if there exists $\alpha \in \mathbb{N}^r$ and $\beta \in \mathbb{N}_+^r$ such that

$$\forall\ k \in \mathbb{N} \quad \alpha + k\beta \in K\ .$$

Remark: In [1] a different definition of extended set has been used. However it is easy to check that both definitions are equivalent.

If $V(\tilde{L})$ is not extended then one can show similarly as in [1] that there there exists a nonregular language in $\{a_1\}^* \ldots \{a_{r-1}\}^*$ fulfilling the assumptions of the proposition. Hence, by the inductive hypothesis we obtain a contradiction. Therefore, we can assume that $V(\tilde{L})$ is extended. Let $\alpha = (\alpha_1, \ldots, \alpha_r)$ and $\beta = (\beta_1, \ldots, \beta_r)$ with $\alpha_1, \ldots, \alpha_r \in \mathbb{N}$ and $\beta_1, \ldots, \beta_r \in \mathbb{N}_+$, be vectors such that

$$\forall\ k \in \mathbb{N} \quad \alpha + k\beta \in V(\tilde{L})\ .$$

Moreover, let $\tilde{M}$ be an ATM which recognizes $\tilde{L}$ in space S. Define the function S' by

$$S'(n) \;:=\; S\Big(\sum_{i=1}^{r} \alpha_i \;+\; n\ \sum_{i=1}^{r} \beta_i\Big)\ .$$

Since $S \in o(\log)$ also $S' \in o(\log)$. Let $\hat{n} := \mathcal{N}_{\tilde{M},S'}$. Then we define

$$\hat{R} := \{a_1^{\alpha_1+(\hat{n}+\ell_1\hat{n}!)\beta_1} \ldots\ a_r^{\alpha_r+(\hat{n}+\ell_r\hat{n}!)\beta_r} \mid \ell_1, \ldots, \ell_r \in \mathbb{N}\} \quad \text{and} \quad \hat{L} := \hat{R} \cap \tilde{L}\ .$$

A contradiction will be obtained from the following claims

Claim 1 $\hat{L}$ can be recognized in constant space.

Claim 2 $\hat{L}$ is nonregular.

Proof of Claim 1: Using for every $i = 1, \ldots, r$ β_i-times the Small-Space-Bound Lemma we obtain that for any sequence of integers $\ell_i \geq 0$

$$\hat{s} \;:=\; Space_{\tilde{M}}(a_1^{\alpha_1+(\hat{n}+\ell_1\hat{n}!)\beta_1} \ldots\ a_r^{\alpha_r+(\hat{n}+\ell_r\hat{n}!)\beta_r}) \;=\; S'(a_1^{\alpha_1+\hat{n}\beta_1} \ldots\ a_r^{\alpha_r+\hat{n}\beta_r})\ . \qquad \text{(i)}$$

Let $\hat{M}$ be an ATM which performs the following algorithm:

Step 1. Check deterministically if the input X has the form $a_1^{\alpha_1+(\hat{n}+\ell_1\hat{n}!)\beta_1} \dots a_r^{\alpha_r+(\hat{n}+\ell_r\hat{n}!)\beta_r}$ for some integers $\ell_1, \dots, \ell_r$.
Reject and stop if this condition does not hold.

Step 2. Move the head to the first symbol of the input and start to simulate the machine $\tilde{M}$.

It is obvious that $\hat{M}$ accepts an input $X = a_1^{\alpha_1+(\hat{n}+\ell_1\hat{n}!)\beta_1} \dots a_r^{\alpha_r+(\hat{n}+\ell_r\hat{n}!)\beta_r}$ iff $\tilde{M}$ accepts X. Hence, we have $L(\hat{M}) = \hat{L}$. It is easy to see that step 1 can be performed within space $O(\log \hat{n}!)$, which is a constant. Moreover from (i) it follows that step 2 also requires only constant space $\hat{s}$. Hence $\hat{M}$ recognizes $\hat{L}$ within constant space. ∎

Proof of Claim 2: A set of the form

$$\{\gamma + (k_1\delta_1, \dots, k_r\delta_r) \mid k_1, \dots, k_r \in \mathbb{N}\}$$

with $\gamma \in \mathbb{N}^r$ and $\delta_1, \dots, \delta_r \in \mathbb{N}$ is called a *grid*. We show that if $\hat{L}$ is regular then there exists an r-dimensional grid in $V(\tilde{L})$.

Assume that $\hat{L}$ is regular. Then, using the pumping lemma for regular languages one can show that there exist integers $\ell \geq 0$ and $\delta_1, \dots, \delta_r > 0$ such that for all $k_1, \dots, k_r \geq 0$

$$\alpha + (\hat{n} + \ell\hat{n}!)\beta + (k_1\delta_1, \dots, k_r\delta_r) \in V(\hat{L}) .$$

Hence, the r-dimensional grid $G := \{\gamma + (k_1\delta_1, \dots, k_r\delta_r) \mid k_1, \dots, k_r \in \mathbb{N}\}$ with $\gamma = \alpha + (\hat{n} + \ell\hat{n}!)\beta$ is a subset of $V(\hat{L})$, which implies $G \subseteq V(\tilde{L})$. From this and the property $V(\tilde{L}) \cap C = \emptyset$ shown above we obtain that $G \cap C = \emptyset$ for the r-dimensional cone C. This yields a contradiction to the following result.

Lemma ([1]) Let $G \subseteq \mathbb{N}^r$ be an r-dimensional grid and let $C \subseteq \mathbb{N}^r$ be an r-dimensional cone. Then $G \cap C \neq \emptyset$. ∎

Recall that a language L is called *strictly nonregular* if there are strings u, v, w, x and y such that $L \cap \{u\}\{v\}^*\{w\}\{x\}^*\{y\}$ is context-free and nonregular. It was shown by Stearns ([20]) that every nonregular deterministic context-free language is strictly nonregular. Therefore, from the proposition above we obtain immediately that if L is a nonregular deterministic context-free, a strictly nonregular language, or a nonregular context-free bounded language, then for ATMs without any bound on the number of alternations it is not possible that L and $\overline{L}$ both belong to $ASpace(o(\log))$. Moreover, from Theorem 7 it follows that the class of languages recognized by space-bounded ATMs with a constant number of alternations is closed under complement. Hence it follows that the language L does not belong to $\bigcup_{k\in\mathbb{N}} \Sigma_k Space(o(\log))$. This completes the proof of Theorem 6.

6 Conclusions

The obvious question remaining is how $\Sigma_1 Space(S)$ and $\Pi_1 Space(S)$ compare. It is somewhat annoying that the techniques developed in this paper do not give any help for the case $k = 1$. It is not completely unrealistic to believe that both classes may be equal, which would give the novel result that a hierarchy is infinite, although its first level collapses.

If one restricts to bounded languages $\Sigma_1 Space(S)$ is closed under complementation and both classes are identical, which has been shown in [2] and [23]. But for $k = 2$ the situation changes completely. The languages $L_{\Sigma 2}$ and $L_{\Pi 2}$ are unary – the most stringent form of a bounded language – and still separate $\Sigma_2 Space(S)$ from $\Pi_2 Space(S)$. Thus a separation of the first level would require a syntatically more complex languages than the second level. For $k > 2$ the languages $L_{\Sigma k}$ and $L_{\Pi k}$ used in this paper to establish the separation are no longer bounded. But by Proposition 4 the third level can also be separated using simple bounded languages $A_{\Sigma 2} \cap B_{\Sigma 2}$ and $A_{\Pi 2} \cup B_{\Pi 2}$ that both are subsets of $\{1\}^*\{0\}\{1\}^*$.

Nothing seems to be known for level 4 and higher. Thus, the sublogartihmic space hierarchy for bounded languages may be even more complex. We have made some observations leading to the conjecture that for bounded languages this hierarchy might indeed consist of only a finite number of distinct levels.

Finally, it would be nice to characterize the exact relationship between co-$\Sigma_k Space(S)$ and $\Pi_k Space(S)$ for sublogarithmic space bounds S and the class of arbitrary languages.

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